Topics in Approximation Algorithms 2006: Final-Term Exam

[Answer all the questions. Total marks: 25 (1 mark for neatness)]

Q1.[3 marks] Give an integer linear program for the minimum spanning tree problem. Assume that you are given a graph $G = (V, E), |V| = n$, with cost function $c : E \to Q^+$.

Q2.[3 marks] We want to compute a maximum weight matching in a bipartite graph $G = (U \cup V, E)$ with a nonnegative edge weight function $w : E \to Q^+$. Suppose we have a matching $M \subseteq E$ and a nonnegative function Π on the vertex set $U \cup V$ such that:
(i) for every vertex x that is not matched by M we have $\Pi(x) = 0$
(ii) for every edge (u, v) we have $\Pi(u) + \Pi(v) \geq w(u, v)$
(iii) for every edge $(u, v) \in M$ we have $\Pi(u) + \Pi(v) = w(u, v)$
Then show that M is a maximum weight matching.

Q3.[2 + 2 + 2 marks]
(a) Give an integer linear program for the shortest path problem betwee two given vertices s and t in a graph $G = (V, E)$ with weight function $w : E \to Q^+$.

(b) We are given a complete directed graph $G = (\{1, 2, \ldots, n\}, E)$ where $E = \{1, 2, \ldots n\} \times \{1, 2, \ldots n\}$ (between every pair of vertices i and j, there is a directed edge from i to j and another directed edge from j to i. Edge (i, j) has a nonnegative cost c_{ij}. Does the following integer LP compute a minimum cost cycle visiting every vertex exactly once (a min TSP tour)? Why/why not?

$$\text{Minimize} \sum_{(i,j)\in E} c_{ij}x_{ij} \text{ subject to}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, \ldots, n$$

$$x_{ij} = 0 \textit{ or } 1 \quad (i, j) \in E$$

(c) Does the following LP compute a maximum weight matching in a non-bipartite graph $G = (V, E)$?

$$\text{Maximize} \sum_{e\in E} x_e \text{ subject to}$$

$$\sum_{e:e \text{ incident on } v} x_e \leq 1 \quad v \in V$$

$$x_e \geq 0 \quad e \in E$$

Q4.[6 marks] Given an undirected graph $G = (V, E)$ with nonnegative edge weights, we wish to remove a minimum weight set of edges so that the resulting graph is bipartite. Obtain an $O(\log m)$ approximation algorithm for this problem where m is the number of edges.

Q5.[6 marks] Given m equations over n variables taking values in $Z_2 = \{0, 1\}$, we want to find an assignment for the variables that maximizes the number of satisfied equations. Give a factor 1/2 randomized algorithm for it, and derandomize it using the method of conditional expectation. (Note that arithmetic here is modulo 2.)