Topics in Graph Theory-2006, Exam-1, 6th March 2006

1. *(from the material covered in class)* Give Erdo's argument for Littlewood-Offord problem with real numbers: Let $r_1, r_2, \cdots, r_n$ be real numbers where $|r_i| \geq 1$ and let $S_1, S_2, \cdots, S_k$ be distinct subsets of $[n]$ such that $|\sum_{t \in S_i} r_t - \sum_{t \in S_j} r_t| < 1$, for $1 \leq i < j \leq k$. Then $k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. **10-marks**

2. *(based on material from class: slightly indirect)* Let the vertices of a hypercube be numbered from 0 to $2^d - 1$ such that two vertices u and v are adjacent if and only if the corresponding bit representations (using d bits) differ in exactly one bit-position. Let S_i be the set of the last i vertices of H_d. Let $h(j)$ denote the number of 1's in the binary representation of i. Then relate sum of $h(j)$'s for the last i numbers j (i.e. $\sum_{2^d - 1 \geq j \geq 2^d - i} h(j)$) with the number of edges in the induced subgraph on S_i. Prove your answer. **10-marks**

3. *(new:)* Let $G = (V, E)$ be a graph and let $f : V \rightarrow \{1, 2, \cdots, n\}$ (where $|V| = n$) be an ordering of its vertices. For $1 \leq k < n$, let I_k (respectively I'_k) denote the first (resp. last) k vertices with respect to this ordering: That is, $I_k = \{f^{-1}(1), f^{-1}(2), \cdots, f^{-1}(k)\}$ and $I'_k = \{f^{-1}(n), f^{-1}(n-1), \cdots, f^{-1}(n-k+1)\}$. Suppose we have the following information: "For any two non-empty subsets $A_1, A_2 \subset V$, with $|A_1| = k_1$ and $|A_2| = k_2$, $d(A_1, A_2) \leq d(I_{k_1}, I'_{k_2})$ where $d(X, Y)$ is defined as the shortest distance between the set X and Y. That is, $d(X, Y) = \min_{(u,v)} d(u, v)$ where $u \in X$ and $v \in Y$. " Now use this information to prove the following. For nay non-empty $A \subset V, |\partial(A)| \geq |\partial(I_k)|$, where $k = |A|$. (Recall that the $\partial(A)$ is the vertex boundary of A: i.e. $A \cup N(A)$.) **10-marks**

4. *(new:)* Let $X = \{1, 2, \cdots, n\}$ and $\mathcal{P}(X)$ be the power set of X. Let $f : \mathcal{P}(X) \rightarrow \{1, \cdots, 2^n\}$ be the simplicical ordering discussed in the last class: i.e. for $A, B \in \mathcal{P}(X), f(A) < f(B)$ iff either $|A| < |B|$ or $|A| = |B|$ AND $\min(A \Delta B) \in A$ (i.e. A is ealier than B with respect to the lexicographic ordering.) Now suppose that two subsets $X, Y \in \mathcal{P}(X)$ are such that $X = \overline{Y}$ and $f(X) - f(Y) = 1$ (i.e. they are consecutive with respect to the simplicial ordering.) How many such pairs (X, Y) are possible for a given n ? List all such pairs for a given n. **10-marks**

5. *(new:)* $I = (V, E)$ is an interval graph if and only if there exists a function Π that maps each vertex $u \in V$ to a closed interval of the form $[l(u), r(u)]$ on the real line such that $(u, v) \in E(I) \iff \Pi(u) \cap \Pi(v) \neq \emptyset$. (We may call Π, an interval representation of $I = (V, E)$.) Show that for $1 \leq i < n, \omega(I) \geq b_v(I, i) + 1$ where $\omega(I)$ is the clique number (i.e. number of vertices in the maximum clique) of the interval graph I. **10-marks**

6. *(new:)* Let H_d be the d-dimensional hypercube. Show that for any integer i where $2^{d-2} \leq i \leq 2^{d-1}, b_e(i, H_d) \geq 2^{d-1}$. **15-marks**