Topics in Graph Theory-2006, Final Exam, 23rd April 2006

For all the questions, assume $G = (V, E)$ is a simple, finite, undirected graph, and $n = |V|$. Let T_d be the complete binary tree of depth d. The minimum number of edges going out of a subset S, where the minimum is taken over subsets of cardinality i, is denoted by $b_e(i, G)$ as usual. We use A to denote the adjacency matrix of the concerned graph, as should be clear from the context.

1. **(Direct:)** Let L be the Laplacian matrix of G, i.e. $L = D - A$, where D is the diagonal matrix with the (i, i)th entry being the degree of the ith vertex and other entries 0. Show that if X is a (column) vector such that $X^T = (x_1, x_2, \cdots, x_n)$, (where $n = |V|$) then,

$$X^T L X = \sum_{(i,j)\in E} (x_i - x_j)^2$$

10-marks

2. **(Direct:)** Let $N(X) = \{u \in V : \exists v \in X, \text{suchthat}, (u, v) \in E\}$. Assuming that G is a k-regular graph, show that $\|Af\|^2 \geq \frac{k^2|X|^2}{|N(X)|}$. where f is the support vector for the subset X, i.e, there is a 1 if the vertex corresponding to a component position belongs to X, otherwise 0. **10-marks**

3. **(Direct:)** Show that for any bipartite graph G, the spectrum of the adjacency matrix is symmetric with respect to the origin: i.e., λ is an eigen value if and only if $-\lambda$ is also an eigen value. **10-marks**

4. Let δ be the minimum degree of G and γ be the cardinality of the minimum cut (i.e., the edge connectivity.) Show that if the second smallest eigen value μ of the Laplacian matrix of G is $> 1 + \frac{\delta}{n-\delta}$, then every minimum cut $(S, \overline{S})$ is such that either S or $\overline{S}$ is a singleton set. (Recall that any cut can be written as $(S, \overline{S})$ for some $S \subset V$). **20-marks**

   (Hint: If not, how big should $\min(|S|, |\overline{S}|)$ be ? Along with this information try to use the theory of Raleigh quotient, you studied in the class.)

5. For a graph $G = (V, E)$, let $ip(G)$, the isoperimetric peak of G be defined as :

$$ip(G) = \max_{1\leq i\leq n} b_e(i, G)$$

   A subdivision operation on edge (u, v) is defined as the combination of the following three operations: (1) Remove edge (u, v), (2) Add a new vertex w, (3) Add two new edges (u, w) and (w, v). In other words, to subdivide an edge (u, v) means to replace (u, v) with the path (u, w, v). If G' can be obtained from G by a series of subdivision operations from G (i.e. $G_1 = G, G_2, \cdots, G_t = G'$ is a sequence of graphs where G_{i+1} is obtained by subdividing an edge of G_i) then G' is homeomorphic to G. Let T_d be the complete binary tree of depth d. Then show that there exists a T' homeomorphic to T_d such that for all induced subgraphs H of $T', ip(H)$ is atmost a constant. (The constant can be taken to be 3 if you wish.) **30-marks**

6. For $G = (V, E)$, let f be an ordering of V i.e., $f : V \to \{1, 2, \cdots, n\}$. Then, let wirelength of $f, wl(f) = \sum_{(i,j)\in E} |f(i) - f(j)|$. Let wirelength of G be $wl(G) = \min_f wl(f)$, where the minimization is done over all possible orderings of V.

   Let T_d be the complete binary tree with depth d. Then show that $wl(T_d) = \theta(2^d d)$. That is, show that there exists, constants c_1, c_2 such that $c_1 2^d d \leq wl(T_d) \leq c_2 2^d d$. **30-marks**

   (Hint: You may have to relate wirelength to an isoperimetric problem. You may use what you have learned in the class freely, but state what you are using.)

7. Let G_1 be a clique on the set $U_1 = \{u_1, u_2, \cdots, u_m\}$ of m vertices. Similarly let G_2 be a clique on the set $U_1 = \{v_1, v_2, \cdots, v_m\}$. Now construct a new graph G' on $2m$ vertices by connecting these two cliques together by adding an edge from u_i to v_i for each $i, 1 \leq i \leq m$. Write down the spectrum of this graph. **30-marks**

   (Hint: Formulate the adjacency matrix of this graph in a natural, but nice way. Then use the following theorem: If $B = PAP^{-1}$, where A, P, B are $n \times n$ matrices and P^{-1} is the inverse of P, then the spectrum of B is the same as that of A. (A and B are called similar matrices.). So try to construct a suitable matrix P so that you can convert the adjacency matrix of the given graph to some thing easier to deal with.)