DAA 2006: Mid-Term Exam II

Q1.[5] Show that for every maximum flow there is a saturated cut which proves that the flow is maximum.

Q2.[5] We have n vectors $v_1, \cdots, v_n$. Each vector v_i is an array $(a_{i1}, \cdots, a_{in})$ of n integers. Compute all $\binom{n}{2}$ dot products (v_i, v_j) for $i \neq j$ in $o(n^3)$ time. [$o(n^3)$ means a function $f(n)$ such that $\lim_{n\to\infty} \frac{f(n)}{n^3} = 0$. An algorithm with running time $O(n^{3-\delta})$ for some $\delta > 0$ is an $o(n^3)$ algorithm.]

Q3.[7.5] Show that the number of distinct minimum cuts in an undirected graph (assume all edge weights are one) is at most $\frac{n(n-1)}{2}$. That is, show that the number of distinct cuts whose value is equal to the value of the min cut in the graph is at most $\frac{n(n-1)}{2}$.

Q4.[7.5] Consider an undirected graph $G = (V, E)$ with $|V| = n$ and $|E| = m$ and let all edge weights be one. For any two vertices u, v let $d(u, v)$ denote the length of the shortest path between u and v in G. Let $S \subset V$ be a set of marked vertices. For any $u \in V$ define

$$d(u, S) = \min_{v \in S} d(u, v).$$

So $d(u, S) = 0$ for all $u \in S$. Give and $O(m + n \log n)$ time algorithm (i.e., the running time of Dijkstra's algorithm which is used to compute shortest paths from one fixed vertex to all other vertices) to compute *all* the values $d(u, S)$ for all $u \in V - S$. That is, in time $O(m + n \log n)$ compute all the $n - |S|$ values $d(u, S)$, where $u \in V - S$.