DAA 2006: Final Exam

Q1.[15]Recall the preflow-push algorithm using the FIFO rule. The number of relabes is $O(n^2)$ and using the FIFO rule, the number of pushes is $O(n^3)$. Show that the *running time* of the entire preflow-push algorithm using the FIFO rule (that is, using the above bounds on relabels and pushes) is $O(n^3)$.

Q2.[10] Give an $O(\log^2 n)$ Monte Carlo algorithm to determine if a given number $n > 1$ is prime or composite where you are promised that n is either prime or a composite non-Carmichael number. (A Carmichael number is a composite number n such that for all $a \in Z_n^*, a^{n-1} = 1 \pmod{n}$.) You can assume that arithmetic on $O(\log n)$ bit numbers takes $O(\log n)$ time.

Q3.[10] Show that if M is any integer such that $0 < M < p_1 p_2 \cdots p_k$ and $p_1 < p_2 < p_3 < \cdots < p_{2k}$ are primes, then for any p choosen uniformly at random form $\{p_1, \cdots, p_{2k}\}$, we have $M \neq 0 \pmod{p}$ with probability at least 1/2.

Q4.[10] Consider the *polynomial product verification* problem: given polynomials $P_1(x), P_2(x), P_3(x) \in F[x]$, where F is a finite field, verify that $P_1(x) \times P_2(x) = P_3(x)$. Assume that the degrees of $P_1(x)$ and $P_2(x)$ are at most n and the size of F is much larger than $4n$ and that arithmetic on elements in F takes unit time. Give an $O(n)$ Monte Carlo algorithm for verifying this polynomial product.

Q5.[15] We have an undirected graph $G = (V, E)$ with edge weights equal to 1. We have a set $S \subseteq E$ of marked edges. Give an efficient algorithm to compute a shortest cycle C in G such that C contains an odd number of marked edges.

(Hint: It is enough to show a transformation of this problem to a shortest paths problem and then invoke a shortest paths algorithm as a block box - there is no need to explicitly describe a shortest paths algorithm.)

Q6.[15] We want to relate the half-plane intersection problem to the convex hull problem in the plane. The convex hull problem asks for a set of n points $P = \{p_1, \cdots, p_n\}$ in the plane and computes the smallest convex set containing these n points. Let us assume that the points in P are such that the origin lies in the interior of the convex hull of P. We can also assume that no 3 points of P lie on the same straight line.

For any point $p = (a, b)$, define the line l_p as $ax + by = 1$. Let the coordinates of p_i be (a_i, b_i) and for each p_i, we can define the corresponding line l_{p_i}. Suppose we are given the n half-planes $\{h_1, \cdots, h_n\}$, where each h_i is of the form $a_i x + b_i y \leq 1$ (that is, one side of the line l_{p_i}). We want to compare teh boundary of $h_1 \cap \cdots \cap h_n$ to the boundary of convex hull of P.

Show that

(a) if p_i lies on the boundary of the convex hull then l_{p_i} has to occur on the boundary of the region $h_1 \cap \cdots \cap h_n$.

(b) if $\overline{p_i p_j}$ is an edge of the convex hull, then the point $l_{p_i} \cap l_{p_j}$ is a vertex of the boundary of the region $h_1 \cap \cdots \cap h_n$.

Q7.[10] Consider two sets A and B, each having n integers in the range from 0 to $10n$. We wish to compute the *Cartesian sum* of A and B, defined by

$$C = \{x + y : x \in A, y \in B\},$$

Note that the integers in C are in the range 0 to $20n$. We want to find the elements in C and the number of times each element of C is realized as a sum of elements in A and B. Show that the problem can be solved in $O(n \log n)$ time.

Q8.[15] Prove that following statements on min cuts: if $(S, V - S)$ is minimum s-t cut in an undirected graph G and $u, v \in S$, then there exists a minimum u-v cut $(S^*, V - S^*)$ such that $S^* \subset S$ or $(V - S^*) \subset S$.

[Use the following property of cuts: if A and B are two subsets of vertices in a graph and $\delta(X)$ represents the size of the cut $(X, V - X)$, then $\delta(A) + \delta(B) \geq \delta(A \cap B) + \delta(A \cup B)$.]