Modeling and Simulation
Test 3 Time: 90 min
Answer all questions

19th Nov, 2005

1. State and justify two algorithms to simulate a non-homogeneous poisson process. 10 marks

2. What can you say about the density function of

$$W_i = \frac{\sum_{j=1}^{i} X_j}{\sum_{j=1}^{n+1} X_j} \quad i = \{1, 2, \ldots, n\}$$

   where $X_i \sim expo(1)$. Where do you think this will be useful for simulation? 10 marks

3. Consider the following counting process

$$N = min\{n : n \geq 2, u_n > u_{n-1}\}$$

   where $u_n \sim U(0, 1)$.

   (a) Compute $E(N)$ and $Var(N)$. 3 marks

   (b) Use anthithetic variates to give another estimator M of $E(N)$ 2 marks

   (c) Let $N = 2 + X$ then show that $N + M = 4 + X$. Find the mean and variance of X. Discuss the reduction of variance of the estimator $\frac{1}{2}(N + M)$ compared to the raw estimator N. 10 marks

4. (a) Prove that 5 marks

$$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$$

   (b) Consider the random variable

$$X = \begin{cases} 1 & \text{if } V_1^2 + V_2^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find $E(X|V_1)$ and show that $E(X|V_1)$ is an estimator of $E(X)$. The two random variable V_1, V_2 are uniformly distributed in [-1, 1]. 5 marks

(c) Compute the percentage reduction of variance of the new estimator compared with the original estimator X. 5 marks

5. (a) Define the hazard rate function $h(t)$, for a continuous non-negative RV X. Assume that it is easy to compute H^{-1}, the inverse function of $H(t) = \int_0^t h(\tau)d(\tau)$. Show that 3 marks

$$T = H^{-1}(X) \quad X \sim expo(1)$$

(b) The discrete hazard rate λ_n is the probability that an item fails at age n, where $n = 0, 1, \ldots$. Let X denote a discrete random variable measuring the life time. Express λ_n as a function of the mass function of X. 2 marks

(c) Show that $min\{n|u \leq \lambda_n\}$ simulates X where $0 < u < 1$. 5 marks

(d) Let $\lambda_n = p$ where $0 \leq p < 1$. Give an efficient algorithm to simulate X. 5 marks