Modeling and Simulation
Assignment 1
Due Date: 10th Sep, 2005

1. For any 3 events show that

$$P(A_1 \cup A_2 \cup A_3) = \sum_{i=1}^{3} P(A_i) - \sum_{i \leq j}^{3} P(A_i A_j) + P(A_1 A_2 A_3)$$

   where P is the probability measure. Generalize this to the case of n events. 5 marks

2. Show that

$$P(A_1 A_2 A_3) = P(A_1) P(A_2 | A_1) P(A_3 | A_2 A_1)$$

   where $P(A_2 A_1)$ is not zero. Generalize this to the case of n events. 5 marks

3. Consider throwing a dice N times. Let p_i be the probability that the face i shows up in a single throw. See that $\sum_{i=1}^{6} p_i = 1$. Define the RV $X_i = N_i$ where N_i is the number of times the face i shows up in N throws. What is the joint pmf of $X_1, X_2, \ldots, X_6$. What is the pmf of X_i. 7 marks

4. Let $Y = X^2$ where X is a continuous RV with density fucntion $f_X(x)$. What is the density function of Y. 5 marks

5. Let X_1 and X_2 be i.i.d continuous RVs with density function $f(x)$. What is the density function of $Z = X_1 - X_2$. 5 marks

6. Consider a continuous RV with density function f which satisfies

$$f(x_0 + x) = f(x_0 - x)$$

   for all x where x_0 is a constant. The function f is then said to be symmetric about x_0. Find the expectation of such a RV. 5 marks

7. For what value of k is $P(X = k)$ is maximized when X is a Binomial RV. 5 marks

8. Let X be $U(0,1)$. What is the cdf of $Y = X + 4$. 5 marks

9. Show that
$$P(X \geq x) \leq \frac{\sigma^2}{x^2 + \sigma^2}, \quad x \geq 0$$
where X is a RV with $E(X) = 0$ and $Var(X) = \sigma^2$. 10 marks

10. Show that
$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$
where X and Y are any two RVs for which $E(X^2)$ and $E(Y^2)$ exists. 5 marks

11. Define the correlation coefficient
$$\rho = \frac{cov(X,Y)}{\sigma_x.\sigma_y}$$
Use the above problem to show that $-1 \leq \rho \leq 1$. 8 marks

12. Generate a Bernoulli RV as follows:

    If $rand \leq p$ then $X = 1$ else $X = 0$, return X

    where *rand* is the random number generator function in matlab. Consider the following experiment. Call the program N times. Let $X = \sum_{i=1}^{N} X_i$ where X_i is the value of X returned in the i^{th} call.

    (a) Fix $N = 10$ and do the experiment 1000 times with $p = 0.5$. Plot
    $$f_k = \frac{\text{number of times } (X = k)}{N}$$
    versus k. Repeat for $N = 20$, 30, 40. Comment on the nature of the curve.

    (b) Fix $N = 1000$ and repeat the experiment 1000 times. Again plot f_k versus k. Repeat your experiment for $p = 0.01$, 0.1, 0.2, 0.4, 0.8. Comment on your results.

    20 marks

13. Consider the game between Jai and Vijay. In the game Jai tosses a biased coin. If head occurs then Vijay gives 1 Re to Jai and if tail occurs Jai gives 1 Re to Vijay. The game is over till either of the two players loses all his money. Simulate this game with the program described in the previous question. Assume that Jai has a Rs, Vijay has b Rs, and bias of the coin is p.

    (a) Play the game 10,000 times. Let f denote the number of times Vijay loses divided by 10,000. Plot f versus a where $b = ka$ and $p = 0.5$ and $k = 1$. On the same plot also show the plots of f versus a for $k = 10, 100, 1000$.

(b) Repeat the above question for different values of $p = 0.01$, 0.2, 0.4, 0.6, 0.9.

(c) Jai also has the option of choosing p. How should he choose p when $k = 1$ and $k = 1000$ so that Vijay is ruined.

20 marks