Discrete Structures: Exam No. 3

1. Let G be a k-regular bipartite graph with $k \geq 2$. Then show that the vertex connectivity $\kappa(G) \neq 1$. **7 marks**

2. Let G be a graph on n vertices. Then show that $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$. **8 marks**

3. (a) Let d and n be such that $d|n$ (i.e. d divides n). Let G be a group of order n. Let $x \in G$ be a generator of G. That is $\{x^0, x^1, x^2, \cdots, x^{n-1}\} = G$. Let

   $$Y_d = \{x^i : \text{ such that order of } x^i \text{ equals } d\}$$

   Show that $|Y_d| = \phi(d)$, where $\phi(d)$ is the number of positive integers $< d$ that are relatively prime to d. **8 marks**

   (b) Show that $\sum_{d|n} \phi(d) = n$. (Note: the summation is over all positive integers $d \leq n$ which divides n.) **7 marks**

4. Let G be a graph defined as follows: its vertex set is the set of all r-element subsets of $\{1, 2, \cdots, n\}$, where $r < n$. Two vertices i and j corresponding to subsets S_i and S_j are adjacent if and only if $S_i \cap S_j = \phi$. Show that $\chi(G) \leq n - 2r + 2$. **10 marks**

5. Let a permutation of $1, 2, \cdots, n$ be selected uniformly at random. Recall that corresponding to a permutation π, a directed graph can be defined where there is a directed edge from vertex i to j if and only if the $\pi(i) = j$. This directed graph is a collection of directed cycles. (The length of the cycles ranges from 1 to n.) Consider the cycle which contains the vertex 1. Define an indicator random variable X_i as follows: $X_i = 1$ if the cycle containing the vertex 1 is of length $\geq i$. Otherwise let $X_i = 0$. Then

   (a) What quantity does $X = X_1 + X_2 + \cdots + X_n$ correspond? **3 marks**

   (b) Find the expectation of X. **7 marks**

6. Consider $K_{\frac{n}{2},\frac{n}{2}}$, the complete bipartite graph on n vertices with both sides having $\frac{n}{2}$ vertices each. With each vertex v, a list of colors $S(v)$ is associated where $|S(v)| > log\ n$. Show that there exists a valid vertex coloring c of G such that the color $c(v)$ assigned to each vertex v belongs to its list $S(v)$, i.e. $c(v) \in S(v)$. **10 marks**

   (Hint: Let A and B be the two sides of the bipartite graph. Let $X = \bigcup_{u \in A} S(u)$. Let Y be a random subset of X which is formed by choosing each color of X with probability $\frac{1}{2}$. Now try to color the vertices on A side using only colors from Y.)