Discrete Structures: Exam No. 2

1. Show that if E_1, E_2, E_3 are mutually independent events, then so are $\overline{E_1}, \overline{E_2}, \overline{E_3}$. **5 marks**

2. Let (Ω, P) be a finite probability space in which all sample points (i.e. elementary events) have the same probability. Show that if $|\Omega|$ is a prime number, then no two nontrivial events E_1 and E_2 can be independent. (An event is nontrivial if it is distinct from ϕ and Ω.) **5 marks**

3. For a permutation $\pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$, "$i$" is a fixed point of π if $\pi(i) = i$. Now, consider picking up a permutation uniformly at random from the set of $n!$ possible permutations of $1, 2, \cdots, n$. Then, what is the expected number of fixed points in the random permutation? **5 marks**

4. Show that $\binom{2n}{n} = \theta(\frac{2^{2n}}{\sqrt{n}})$. (In other words, show that there exists constants c_1 and c_2 such that $c_1 \cdot (\frac{2^{2n}}{\sqrt{n}}) \leq \binom{2n}{n} \leq c_2 \cdot (\frac{2^{2n}}{\sqrt{n}})$.) **5 marks**

5. Suppose $n \geq 4$ and H be an n-uniform hypergraph with at most $\frac{4^{n-1}}{3^n}$ hyper edges. Prove that there is a coloring of the vertices of H by 4 colors so that in every hyper edge all four colors are represented. **5 marks**

6. How can you use a fair coin to select a fruit uniformly from three fruits, say apple, orange and banana? (A fair coin is one which gives head with probability $\frac{1}{2}$ and tail with probability $\frac{1}{2}$.) **7 marks**

7. Given a biased coin with unknown bias (i.e. head with probability p, and tail with probabilty $1-p$, where p is not known), how can we use it to simulate a fair coin? **8 marks**

8. Let $\mathcal{F}$ be a family of subsets of $\{1, 2, \cdots, n\}$ such that there is no pair of sets A and B in $\mathcal{F}$ satisfying $A \subset B$. (Let such a family $\mathcal{F}$ be called an antichain.) Then show that:

$$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1$$

where f_k is the number of sets of cardinality k in $\mathcal{F}$. Using the above inequality infer that, for any antichain $\mathcal{F}$

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

(Hint: Choose a random permutation of $1, 2, \cdots, n$. A possible indicator variable you can define is X_k, where $X_k = 1$ if the first k numbers in your permutation yield a set in $\mathcal{F}$.) **10 marks**

9. Let $N = \{1, 2, \cdots, n\}$, and let $U = N \times N \times \cdots \times N = N^d$, where n, d are sufficiently large integers. Let a combinatorial rectangle C be defined as $C = R_1 \times R_2 \times \cdots \times R_d$, where $\phi \neq R_i \subseteq N$. The volume of a combinatorial rectangle C is defined as $vol(C) = \frac{|C|}{|U|} = \frac{|C|}{n^d}$. An ϵ-hitting set is defined as a subset $S \subseteq U$, such that for every combinatorial rectangle C with $vol(C) \geq \epsilon, |C \cap S| \geq 1$. Assume that $\frac{1}{n^d} \leq \epsilon \leq \frac{1}{2}$. Show that if S is a ϵ-hitting set, then $|S| \geq \frac{1}{2\epsilon}$.

Hint: Note that for some ϵ is the given range, it may not be even possible to find any combinatorial rectangle of that volume. So, try to design a random experiment, by which you can choose a combinatorial rectangle of volume, say in the range ϵ and 2ϵ, by randomly selecting subsets $R_1, R_2, \cdots, R_d$ of N, and then taking the cross product of these sets. You have to give rigorous arguments to justify whatever you do. **10 marks**