Discrete Structures: Exam No. 1

1. Let G be a graph obtained from a complete graph by deleting just one edge. Show that the edges of G can be oriented in such a way that the statement of the Gallai-Milgram theorem becomes tight for this orientation of G. **5 marks**

2. Consider a tournament. (that is a complete graph with an orientation). Then show that there exists a vertex v such that from every other vertex u, there exists a path of length at most 2 to v. (Hint: Consider a vertex with maximum incoming degree). **5 marks**

3. Prove that for any graph G, $\alpha(G) + match(G) \leq n \leq \alpha(G) + 2 \cdot match(G)$ where $\alpha(G)$ is the independence number, $match(G)$ is the cardinality of the maximum matching, and n is the number of vertices in G. **5 marks**

4. Let $f : \{1, 2, \cdots, n\} \rightarrow \{1, 2, \cdots, n\}$ be a bijection. Define a graph with vertex set $V = \{1, 2, \cdots, n\}$ and i and j (where $i < j$) are adjacent if and only if $f(i) > f(j)$.

   (a) Consider the following sequence S of numbers

   $$f(1), f(2), \cdots, f(n)$$

   To which graph parameter of G does the longest decreasing subsequence of S correspond to? (Note: if 1,4,5,3,2 is a sequence of numbers, then 4,3,2 is a decreasing subsequence of this sequence). To which parameter does the longest increasing subsequence correspond to? (Prove your answers). **5 marks**

   (b) Use Dilworth's theorem to show that in the sequence $f(1), \cdots, f(n)$, either there exists an increasing subsequence of length at least $\sqrt{n}$ or there exists a decreasing subsequence of length at least $\sqrt{n}$. (The length of a subsequence is the number of terms in the subsequence, for example the subsequence 1,4,5 has length 3). **10 marks**

   (c) (extra) Show that G is a compatibility graph. (In other words, show that G has a transitive orientation.)

5. Let G be a graph with n nodes and maximum degree Δ. Then show that $\alpha(G) \geq \frac{n}{\Delta+1}$. **10 marks**

6. Let G be a connected graph having an even number of vertices. Show that if $\alpha(G) \leq 2$, then G has a perfect matching. **10 marks**

7. Let G be a bipartite graph. Let $L(G)$ denote its line graph. Then show that $\alpha(L(G)) = k(L(G))$ where $k(L(G))$ is the clique cover number of $L(G)$ (i.e. the minimum number of cliques required to cover all the vertices of $L(G)$, and $\alpha(G)$ is the independence number. **10 marks**