Selection Lemmas for various geometric objects

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Abstract

Selection lemmas are classical results in discrete geometry that have been well studied and have applications in many geometric problems like weak epsilon nets and slimming Delaunay triangulations. Selection lemma type results typically show that there exists a point that is contained in many objects that are induced (spanned) by an underlying point set.

In the first selection lemma, we consider the set of all the objects induced (spanned) by a point set P. This question has been widely explored for simplices in $\mathbb{R}^d$, with tight bounds in $\mathbb{R}^2$. In our paper, we prove first selection lemma for other classes of geometric objects. We also consider the strong variant of this problem where we add the constraint that the piercing point comes from P. We prove an exact result on the strong and the weak variant of the first selection lemma for axis-parallel rectangles, special subclasses of axis-parallel rectangles like quadrants and slabs, disks (for centrally symmetric point sets). We also show non-trivial bounds on the first selection lemma for axis-parallel boxes and hyperspheres in $\mathbb{R}^d$.

In the second selection lemma, we consider an arbitrary m sized subset of the set of all objects induced by P. We study this problem for axis-parallel rectangles and show that there exists an point in the plane that is contained in $\frac{m^3}{24n^4}$ rectangles. This is an improvement over the previous bound by Smorodinsky and Sharir [20] when m is almost quadratic.

1 Introduction

Let P be a set of points in $\mathbb{R}^d$. Consider the family of all objects $\mathcal{R}$ of a particular kind (eg. hyperspheres, boxes, simplices, ...) such that each object in $\mathcal{R}$ has a distinct tuple of points from P on its boundary. For example, in $\mathbb{R}^2$, $\mathcal{R}$ could be the family of $\binom{n}{3}$ triangles such that each triangle has a distinct triple of points of P as its vertices. $\mathcal{R}$ is called the set of all objects induced (spanned) by P. Various questions related to geometric objects induced by a point set have been studied in the last few decades. In this paper, we look at

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the problem of bounding the largest subset of $\mathcal{R}$ that can be hit/pierced by a single point.

Combinatorial results on these questions are referred as *Selection Lemmas* and are well studied. A classical result in discrete geometry is the *First Selection Lemma* [8] which shows that there exists a point that is present in $\frac{2}{9} \cdot \binom{n}{3}$ (constant fraction of) triangles induced by P. Bukh [9] provides a simple and elegant proof of the above statement. Moreover, it is known that the constant in this result is tight [10]. Interestingly, both [8, 10] use the centerpoint as the piercing point.

Let P be a set of n points in $\mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is said to be a *centerpoint* of P if any halfspace that contains x contains at least $\frac{n}{d+1}$ points of P. Equivalently, x is a centerpoint if and only if x is contained in every convex object that contains more than $\frac{d}{d+1}n$ points of P. It has been proved that a centerpoint exists for any point set P and the constant $\frac{d}{d+1}$ is tight [18]. The centerpoint question has also been studied for special classes of convex objects like axis-parallel rectangles, halfplanes and disks [2]. Another variant of the centerpoint called strong centerpoint, where the centerpoint is required to be an input point, has also been studied [4].

The first selection lemma has also been considered for simplices in $\mathbb{R}^d$. This is an important result in discrete geometry and it has been used in the construction of weak ϵ-nets for convex objects [16]. Bárány [6] showed that there exists a point $p \in \mathbb{R}^d$ contained in at least $c_d \cdot \binom{n}{d+1} - O(n^d)$ simplices induced from P, where $c_d \geq \frac{1}{(d+1)^d}$. Wagner [21] improved this bound to $c_d \geq \frac{d^2+1}{(d+1)^{d+1}}$. Gromov [13] developed a new topological method which established an improved lower bound of $c_d \geq \frac{2d}{(d+1)!(d+1)}$. Furthermore, Karasev [14] gave a simplified and elegant proof for Gromov's bound and Matousek et al. [17] provided an exposition of the combinatorial components in Gromov's proof. For the upper bound, Bukh et al. [10] showed that there exists a point set in $\mathbb{R}^d$ such that no point is present in more than $(\frac{n}{d+1})^{d+1} + O(n^d)$ induced simplices i.e. $c_d \leq \frac{(d+1)!}{(d+1)^{(d+1)}}$. For $d = 2$, this shows that the bound for c_d is tight. Furthermore they conjectured that this bound was tight for $d \geq 3$. For the case of $\mathbb{R}^3$, Basit et al. [7] improved the lower bound for the first selection lemma in $\mathbb{R}^3$ and showed that there exists a point present in $0.00227 \cdot n^4$ simplices (tetrahedrons) spanned by P i.e. $c_3 \geq 0.05448$. Further improvements on c_3 were shown in [13, 15, 17], with $c_3 \geq 0.07480$ being the best known lower bound [15].

A generalization of the first selection lemma, known as the *Second Selection Lemma*, considers an m-sized arbitrary subset $\mathcal{S} \subseteq \mathcal{R}$ of distinct induced objects of a particular kind and shows that there exists a point which is contained in $f(m, n)$ objects of $\mathcal{S}$. The second selection lemma has been considered for various objects like simplices, boxes and hyperspheres in $\mathbb{R}^d$ [1, 3, 11, 20]. Aronov et al. [3] showed that for any set P of n points and any set T of t triangles induced by P, there exists a point p in the interior of at least $f(t, n) = \frac{t^3}{2^9 n^6 \log^5 n}$, when $t = n^{3-\alpha}, \alpha \leq 1$. Their motivation was to derive an upper bound on the number of halving planes of a finite set of points in $\mathbb{R}^3$. Alon et al. [1] showed that, for any family F of $\alpha\binom{n}{d+1}$ induced simplices, there exists a point contained in at least $c\alpha^{s_d}\binom{n}{d+1}$ simplices of F, where c, s_d are constants.

Chazelle et al. [11] looked at this problem for hyperspheres with the motivation of reducing the complexity of Delaunay triangulations for points in $\mathbb{R}^3$.

They proved a selection lemma for intervals in the line and then extended it for axis-parallel boxes in $\mathbb{R}^d$, by induction on dimension. This in turn was used for the proof of the selection lemma for diametrical spheres induced by a pair of points, by using the fact that any diametrical sphere induced by a pair of points would contain the corresponding induced axis-parallel box. This gave a bound of $\Omega\left(\frac{m^2}{n^2\log^{2d-2}(\frac{n^2}{m})}\right)$ for rectangular boxes in d dimensions (and hence the diametrical hyperspheres as well) and was extended to $\Omega\left(\frac{m^2}{n^2\log^{2d}(\frac{n^2}{m})}\right)$ for general hyperspheres in d dimensions.

Smorodinsky and Sharir [20] improved the bounds obtained in [11] by using a probabilistic proof very similar to the one used in the proof of Crossing lemma [16]. Note that this paper proved that the point which pierced a lot of disks (pseudo-disks) and the d-dimensional hyperspheres came from P. In the case of the axis-parallel rectangles, they proved a lower bound of $\Omega(\frac{m^2}{n^2\log^2 n})$ and an improved upper bound of $O(\frac{m^2}{n^2\log(\frac{n^2}{m})})$. However, in this case the piercing point could be any point in $\mathbb{R}^2$.

As mentioned earlier, first selection lemma has been extensively studied for simplices in $\mathbb{R}^d$. However, no previous work is known on first selection lemma for other geometric objects, to the best of our knowledge. In our paper, we explore the first selection lemma for other geometric objects like axis-parallel boxes and hyperspheres in $\mathbb{R}^d$. We call the case where the piercing point $p \in \mathbb{R}^d$ (same as the previous literature) as the *weak variant*. We also consider the *strong variant* of the first selection lemma where we add the constraint that the piercing point $p \in P$. We prove an exact result on the strong and weak variant of the first selection lemma for axis-parallel rectangles, quadrants, slabs and disks (for centrally symmetric point sets). Note that the first selection lemma for triangles [8, 10] used the centerpoint as the piercing point to prove exact bounds. Interestingly, we also use the strong and weak centerpoint for the respective objects to prove our results in sections 2, 3 and 5.

Let P be a set of n points in $\mathbb{R}^d$ in general position i.e., no two points have the same coordinate in any dimension and no $d+2$ points lie on the same hypersphere. Let $\mathcal{F}$ be a family of objects induced by P. For any point p, let $\mathcal{F}_p \subseteq \mathcal{F}$ be the set of objects that contain p and $f_p^{\mathcal{F}} = |\mathcal{F}_p|$. Let $s^{\mathcal{F}}(n)$ and $w^{\mathcal{F}}(n)$ denote the bounds for the strong and the weak variant of the first selection lemma for a family of objects $\mathcal{F}$. In particular,

$$s^{\mathcal{F}}(n) = \min_{P,|P|=n} (\max_{p\in P} f_p^{\mathcal{F}})$$
$$w^{\mathcal{F}}(n) = \min_{P,|P|=n} (\max_{p\in \mathbb{R}^d} f_p^{\mathcal{F}})$$

Our results for the first selection lemma for various families of objects are summarized in Table 1.

Family of Objects $\mathcal{F}$	**Dim**	$s^{\mathcal{F}}(n)$		$w^{\mathcal{F}}(n)$	
		Lower Bound	Upper Bound	Lower Bound	Upper Bound
Axis-parallel rectangles	2	$n^2/16$		$n^2/8$	
Axis-parallel boxes	d	-		$\frac{n^2}{2^{(2^d-1)}}$	$\frac{n^2}{2^{(d+1)}}$
Orthants	2	$n^2/4$		$n^2/2$	
Axis-parallel slabs	2	$3n^2/8$		$n^2/2$	
Skylines	2	$n^2/9$	$n^2/8$	$n^2/4$	
Disks	2	$n^2/16$	$n^2/9$	$n^2/6$	$n^2/4$
Disks (Centrally Symmetric Point Sets)	2	$n^2/8$		$n^2/4$	
Hyperspheres	d	-		$\frac{n^2}{2(d+1)}$	$n^2/4$
Hyperspheres (Centrally Symmetric Point Sets)	d	-		$n^2/4$	

Table 1: First selection lemma Bounds for various families of objects

We next consider the second selection lemma for axis-parallel rectangles in $\mathbb{R}^2$. We prove that there exists a point $p \in \mathbb{R}^2$ that is contained in at least $\frac{m^3}{24n^4}$ axis-parallel rectangles of $\mathcal{S}$. This bound is an improvement over the previous bound in [11, 20] when $m = \Omega(\frac{n^2}{\log^2 n})$. We use an elegant double counting argument to obtain this result.

In section 2, we prove exact results for strong and weak variants of first selection lemma for axis-parallel rectangles. Section 3 proves tight or almost tight bounds for the strong and weak variants of first selection lemma for families of special rectangles like orthants, slabs and skylines. In section 4, we prove bounds for the weak variant of first selection lemma for boxes in $\mathbb{R}^d$. In section 5, we prove bounds for the strong variant of first selection lemma for induced disks in $\mathbb{R}^2$ and prove bounds for the weak variant of first selection lemma for hyperspheres in $\mathbb{R}^d$. Section 6 proves improved bounds for second selection lemma for axis-parallel rectangles.

2 Rectangles

In this section, we prove the first selection lemma for axis-parallel rectangles. Let $R(u, v)$ be the axis-parallel rectangle induced by u and v where $u, v \in P$ i.e., $R(u, v)$ has u and v as diagonal points. Let $\mathcal{R}$ be the set of all induced axis-parallel rectangles $R(u, v)$ for all $u, v \in P$. Let p be any point and v and h be the vertical and horizontal lines passing through p, dividing the plane into four quadrants as shown in figure 3. Let $|A|$ represent $|A \cap P|$ (similar for all quadrants). $\mathcal{R}_p$ consists of exactly those rectangles which are induced by a pair of points present in diagonally opposite quadrants.

2.1 Weak variant

In this section, we obtain tight bounds for $w^{\mathcal{R}}(n)$.

Theorem 1. $w^{\mathcal{R}}(n) = \frac{n^2}{8}$.

Proof. Let p be the weak centerpoint for rectangles [2]. We claim that $f_p^{\mathcal{R}} \geq \frac{n^2}{8}$.

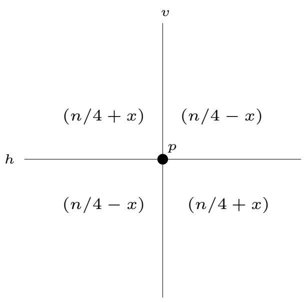


Figure 1: Lower bound

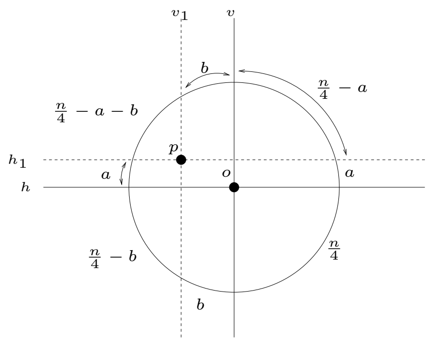


Figure 2: Upper bound construction

Let p divide the plane into four quadrants as shown in figure 1. W.l.o.g let the top left quadrant contain $(\frac{n}{4} + x)$ points. Therefore, the remaining points are distributed among the three other quadrants. Then,

$$\begin{aligned} f_p^{\mathcal{R}} &= \left(\frac{n}{4} - x\right)^2 + \left(\frac{n}{4} + x\right)^2 \\ &= 2 \cdot \left(\frac{n^2}{16}\right) + 2 \cdot x^2 \end{aligned}$$

Thus, $f_p^{\mathcal{R}} \geq \frac{n^2}{8}$. Therefore, $w^{\mathcal{R}}(n) \geq \frac{n^2}{8}$.

For the upper bound, consider a set P of n points uniformly arranged along the boundary of a circle. Let h and v be horizontal and vertical lines that bisect P, intersecting at o. W.l.o.g, let p be any point inside the circle in the top left quadrant and let h_1 and v_1 be the horizontal and vertical lines passing through p. Let a be the number of points from P below h_1 that is present in the top left quadrant defined by h and v. Similarly, let b be the number of points from P to the right of v_1 that is present in the top left quadrant defined by h and v. The number of points in each of the four quadrants defined by h_1 and v_1 is as shown in figure 2.

$$\begin{aligned} f_p^{\mathcal{R}} &= \left(\frac{n}{4} - b + a\right) \cdot \left(\frac{n}{4} - a + b\right) + \left(\frac{n}{4} - a - b\right) \cdot \left(\frac{n}{4} + a + b\right) \\ &= \frac{n^2}{8} - 2(a^2 + b^2) \end{aligned}$$

Since $a, b \geq 0$, $f_p^{\mathcal{R}} \leq \frac{n^2}{8}$ for all points $p \in \mathbb{R}^2$. Therefore, $w^{\mathcal{R}}(n) \leq \frac{n^2}{8}$. □

2.2 Strong variant

In this section, we obtain exact bounds for $s^{\mathcal{R}}(n)$.

Theorem 2. $s^{\mathcal{R}}(n) = \frac{n^2}{16}$

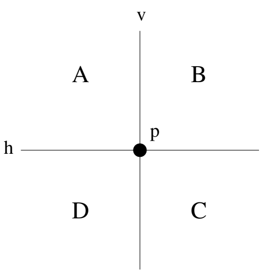


Figure 3: Lower bound

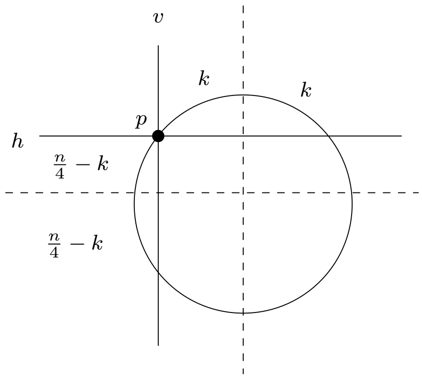


Figure 4: Upper bound construction

Proof. Let p be the strong centerpoint of P w.r.t axis-parallel rectangles. Then any axis-parallel rectangle that contains more than $\frac{3n}{4}$ points from P contains p [4]. We claim that p is contained in at least $\frac{n^2}{16}$ rectangles from $\mathcal{R}$.

Let p partition P into four quadrants as shown in figure 3. If $|A|, |C| \geq \frac{n}{4}$, then p is contained in at least $\frac{n^2}{16}$ rectangles from $\mathcal{R}$. Therefore, assume $|A| = \frac{n}{4} - x$. Now, there are two cases.

Case 1. $|C| \leq \frac{n}{4}$: W.l.o.g, assume that $|C| = \frac{n}{4} - y$ and $x \geq y$. Therefore $|B \cup D| = \frac{n}{2} + x + y$. The value of $f_p^{\mathcal{R}}$ is minimized when the value of $|B| \times |D|$ is minimized. Since $|A| = \frac{n}{4} - x$ and there can be at most $\frac{3n}{4}$ points on either sides of h and v, both B and D contain at least x points. Therefore, $f_p^{\mathcal{R}}$ is minimized when $|B| = \frac{n}{2} + y$ and $|D| = x$. Then,

$$\begin{aligned} f_p^{\mathcal{R}} &\geq \left(\frac{n}{4} - x\right)\left(\frac{n}{4} - y\right) + \left(\frac{n}{2} + y\right)x \\ &\geq \frac{n^2}{16} \end{aligned}$$

Case 2. $|C| > \frac{n}{4}$: Assume $|C| = \frac{n}{4} + y$. Therefore $|B \cup D| = \frac{n}{2} + x - y$. By similar reasons as in case 1, the value of $f_p^{\mathcal{R}}$ is minimized when $|B| = \frac{n}{2} - y$ and $|D| = x$. Therefore,

$$\begin{aligned} f_p^{\mathcal{R}} &\geq \left(\frac{n}{4} - x\right)\left(\frac{n}{4} + y\right) + \left(\frac{n}{2} - y\right)x \\ &\geq \frac{n^2}{16} - 2xy + \frac{n}{4}(x + y) \end{aligned}$$

The value of $f_p^{\mathcal{R}}$ is minimized when $\frac{n}{4}(x+y) - 2xy$ is minimized. Since $|A \cup B| \geq \frac{n}{4}$ and $x + y \leq \frac{n}{2}$, this value is minimized when $x = y = \frac{n}{4}$. Thus, $s^{\mathcal{R}}(n) \geq \frac{n^2}{16}$.

For the upper bound, consider a set P of n points arranged uniformly along the boundary of a circle as in figure 4. Now, we claim that any point $p \in P$ is contained in at most $\frac{n^2}{16}$ rectangles of $\mathcal{R}$. W.l.o.g, let p be a point in the top left quadrant of the circle that is k points away from the topmost point in P. Let h and v be the horizontal and vertical lines passing through p. h and v divide the plane into four quadrants. Therefore $f_p^{\mathcal{R}} = (\frac{n}{2} - 2k)2k = nk - 4k^2$. This value is maximized when $k = \frac{n}{8}$. Thus, $s^{\mathcal{R}}(n) \leq \frac{n^2}{16}$.

□

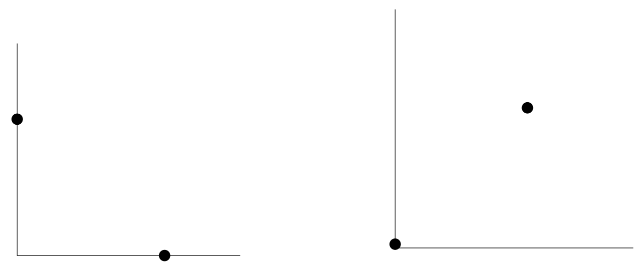
3 Special Rectangles

In this section, we prove bounds for the first selection lemma for some special families of axis-parallel rectangles.

Let p be any point and v and h be the vertical and horizontal lines passing through p, dividing the plane into four quadrants as shown in figure 3. Let $|A|$ represent $|A \cap P|$(similar for all quadrants).

3.1 Quadrants

Quadrants are infinite regions defined by two mutually orthogonal halfplanes. We consider induced quadrants of a fixed orientation as shown in figure 5. If two points are in monotonically decreasing position, then the induced quadrant is defined by two rays passing through the points (see figure 5(a)). Otherwise, the quadrant is anchored at the point with the smaller x and y co-ordinate and the other point is contained in the quadrant (see figure 5(b)). In this case, the same quadrant may be induced by different point pairs. Let $\mathcal{O}$ represent the family of quadrants induced by a point set. Note that the family of all induced quadrants is a multiset.



(a) (b)

Figure 5: Quadrants induced by two points

The weak variant of the first selection lemma is trivial. Let us take the point (x_{max}, y_{max}), where x_{max} and y_{max} are the maximum values of the x and y coordinates of P. It is easy to see that this point is present in all the induced quadrants i.e. $w^{\mathcal{O}}(n) = \frac{n^2}{2}$.

We also prove a tight bound for the strong variant.

Lemma 3. *For any point set P of n points, there exists $p \in P$ such that p is contained in all quadrants that contain more than $\frac{n}{2}$ points from P.*

Proof. Let h be a horizontal line such that it has $\frac{n}{2} - 1$ points of P below it and v be a vertical line that contains $\frac{n}{2} - 1$ points of P to the left of it. h and v divide P into four quadrants as shown in figure 3. By construction, $|B| + |C| = \frac{n}{2} + 1$ and $|C| \leq \frac{n}{2} - 1$. Therefore $B \cap P \neq \emptyset$.

Let $p \in P$ be any point in B. Clearly any quadrant that does not contain p lies completely to the right of p or completely above p and therefore contains at most $\frac{n}{2}$ points. Therefore, any quadrant that contains more than $\frac{n}{2}$ points from P contains p.

□

Theorem 4. $s^{\mathcal{O}}(n) = \frac{n^2}{4}$

Proof. Let $p \in P$ be a point as described in lemma 3 i.e, p is contained in all quadrants that contain more than $\frac{n}{2}$ points from P. We claim that p is contained in at least $\frac{n^2}{4}$ induced quadrants.

Let p divide the plane into four quadrants as shown in figure 3. We know that,

$$|A| + |B| \leq \tfrac{n}{2}$$
$$|B| + |C| \leq \tfrac{n}{2}$$

Assume $|D| = x$. Therefore, $|A|, |C| \geq \frac{n}{2} - x$.

$$\begin{aligned} f_p^{\mathcal{O}} &= \frac{|D|^2}{2} + |D|\left(|A| + |B| + |C|\right) + |A|.|C| \\ &\geq \frac{x^2}{2} + x(n-x) + \left(\frac{n}{2} - x\right)^2 \\ &\geq \frac{n^2}{4} \end{aligned}$$

Therefore p is contained in at least $\frac{n^2}{4}$ induced quadrants.

To prove the upper bound, consider P as n points arranged in a monotonically decreasing order. Let p be any point in P. Then p is contained in all quadrants induced by two points $q, r \in P$ where q lies above p and r lies below p. Let p be x points away from the topmost point in P. Therefore, $f_p^{\mathcal{O}} = x(n-x)$ The value of $f_p^{\mathcal{O}}$ is maximized when $x = \frac{n}{2}$. Therefore $s^{\mathcal{O}}(n) \leq \frac{n^2}{4}$. □

3.2 Axis-Parallel Slabs

Axis-parallel slabs are a special class of axis-parallel rectangles where two horizontal or two vertical sides are unbounded. Each pair of points $p(x_1, y_1)$ and $q(x_2, y_2)$ induces two axis-parallel slabs of the form $[x_1, x_2] \times (-\infty, +\infty)$ and $(-\infty, +\infty) \times [y_1, y_2]$. Let $\mathcal{S}$ represent the family of $2\binom{n}{2}$ axis-parallel slabs induced by P.

We first look at the weak variant for axis-parallel slabs. Let x_{med} be the median of P when the points are projected onto the x axis. Similarly, let y_{med} be the median of P when the points are projected onto the y axis. We claim that (x_{med}, y_{med}) is present in $\frac{n^2}{2}$ induced slabs. Indeed, x_{med} is present in at least $\frac{n^2}{4}$ intervals, obtained by projecting the vertical slabs onto the x axis. Similarly, y_{med} is present in at least $\frac{n^2}{4}$ intervals, obtained by projecting the

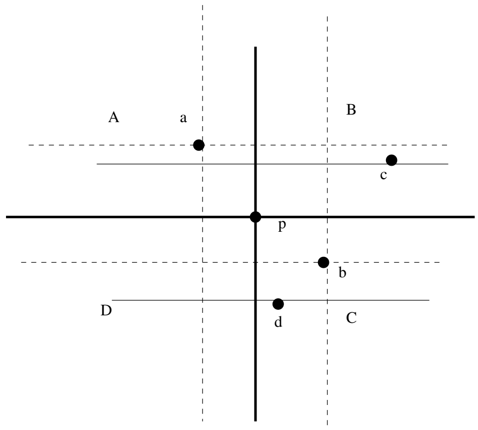


Figure 6: Both slabs defined by a and b contain p, whereas only the horizontal slab defined by c and d contains p

horizontal slabs onto the y axis. Since the set of horizontal and vertical slabs are disjoint, (x_{med}, y_{med}) is present in at least $\frac{n^2}{4} + \frac{n^2}{4} = \frac{n^2}{2}$ induced slabs. It can be easily seen that this bound is tight.

Now we consider the strong variant. Let $p \in P$ be any arbitrary point. Note that for a pair of points $x, y \in P$, p is contained in both the vertical and horizontal axis-parallel slabs induced by them if x and y are present in diagonally opposite quadrants w.r.t p and in exactly one of the induced axis-parallel slabs if x and y are present in adjacent quadrants w.r.t p (see figure 6). Therefore,

$$f_p^{\mathcal{S}} = 2\left(|A|.|C| + |B|.|D|\right) + \left(|A| + |C|\right)\left(|B| + |D|\right).$$

Theorem 5. $s^{\mathcal{S}}(n) = \frac{3n^2}{8}$

Proof. Let $p \in P$ be the strong centerpoint for axis-parallel rectangles [4]. Note that this is also a strong centerpoint for axis-parallel slabs i.e., any axis-parallel slab that contains more than $\frac{3n}{4}$ points from P contains p. We claim that p is contained in at least $\frac{3n^2}{8}$ induced axis-parallel slabs.

Let p divide the plane into four quadrants as shown in figure 3. If $|A| = \frac{3n}{4}$ then $|C| = \frac{n}{4}$ and $f_p^{\mathcal{S}} \geq \frac{3n^2}{8}$. Therefore, assume that $|A| = \frac{3n}{4} - x$. Assume that $x \leq \frac{n}{2}$ (There exists at least one quadrant such that this is true). Now there are two cases:

1. $|C| = \frac{n}{4} - y$:

   Since p is a strong centerpoint, adjacent quadrants have at least $\frac{n}{4}$ points. Therefore quadrants B and D should contain at least y points of P. Also, adjacent quadrants have at most $\frac{3n}{4}$ points. Therefore quadrants B and D have at most x points of P. This implies $x \geq y$.

$$f_p^{\mathcal{S}} = 2\left(|B| \cdot |D| + \left(\frac{3n}{4} - x\right)\left(\frac{n}{4} - y\right)\right) + (x + y)\left(n - (x + y)\right)$$

$f_p^{\mathcal{S}}$ is minimized when $|B|.|D|$ is minimized i.e., the points are distributed as unevenly as possible between B and D. Therefore, $f_p^{\mathcal{S}}$ is minimized when $|B| = x$ and $|D| = y$.

$$\begin{aligned} f_p^{\mathcal{S}} &= 2\left(xy + \left(\frac{3n}{4} - x\right)\left(\frac{n}{4} - y\right)\right) + (x+y)\left(n - (x+y)\right) \\ &= \frac{3n^2}{8} + 2xy + \frac{nx}{2} - \frac{ny}{2} - x^2 - y^2 \\ &= \frac{3n^2}{8} + \left(\frac{n}{2}(x-y) - (x-y)^2\right) \geq \frac{3n^2}{8} \end{aligned}$$

2. $|C| = \frac{n}{4} + y$:
   In this case,

$$f_p^{\mathcal{S}} = 2\left(|B| \cdot |D| + \left(\frac{3n}{4} - x\right)\left(\frac{n}{4} + y\right)\right) + (x-y)\left(n - (x-y)\right)$$

   By reasons similar to case 1, $0 \leq |B|, |D| \leq x$. The value of $f_p^{\mathcal{S}}$ is minimized when B or D is empty. Therefore,

$$\begin{aligned} f_p^{\mathcal{S}} &= 2\left(\frac{3n}{4} - x\right)\left(\frac{n}{4} + y\right) + (x-y)\left(n - (x-y)\right) \\ &= \frac{3n^2}{8} + x\left(\frac{n}{2} - x\right) + y\left(\frac{n}{2} - y\right) \geq \frac{3n^2}{8} \end{aligned}$$

To prove the upper bound, consider P as n points arranged along the boundary of a circle. Let $p \in P$. W.l.o.g assume that p is k points away from the topmost point and $k \leq \frac{n}{4}$. h_p and v_p divides the plane into four regions containing $2k, \frac{n}{2}, \frac{n}{2} - 2k, 0$ points from P. Therefore,

$$\begin{aligned} f_p^{\mathcal{S}} &= 2.2k\left(\frac{n}{2} - 2k\right) + \frac{n}{2}.\frac{n}{2} \\ &= 2nk - 8k^2 + \frac{n^2}{4} \end{aligned}$$

The value of $f_p^{\mathcal{S}}$ is maximized when

$$2n - 16k = 0$$
$$\text{i.e., } k = \tfrac{n}{8}$$

Therefore,

$$f_p^{\mathcal{S}} \leq \tfrac{3n^2}{8}$$

□

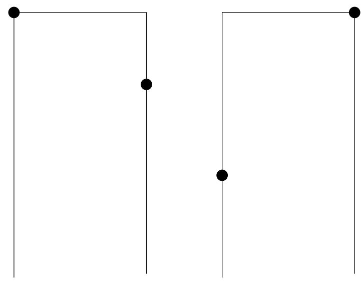


Figure 7: Skyline induced by two points

3.3 Skylines

Skylines are axis-parallel rectangles that are unbounded along a fixed axis, say negative Y axis. A skyline induced by two points has the point with the higher y-coordinate at one corner and the other point in the opposite vertical edge (see figure 7). Let $\mathcal{K}$ represent the family of all $\binom{n}{2}$ skylines induced by P.

As in the case of induced orthants and slabs, the weak first selection lemma for skylines is straightforward. Let x_{med} be the median of P projected onto the x axis. Since the skylines can be assumed to be anchored on the x axis, x_{med} is present in at least $\frac{n^2}{4}$ intervals skylines. This is because x_{med} is present in $\frac{n^2}{4}$ intervals obtained by projecting $\mathcal{K}$ on the x axis. It is easy to see that this bound is tight.

For the strong variant of the first selection lemma, we prove almost tight bounds.

Lemma 6. *For any set P of n points, there exists $p \in P$ such that any skyline that contains more than $\frac{2n}{3}$ points from P contains p.*

Proof. Let v_1(resp. v_2) be a vertical line that has $\frac{n}{3} - 1$ points of P to the left(resp. right) of it. Let h be a horizontal line that has $\frac{n}{3}$ points of P above it. Thus we get a grid-like structure as shown in figure 8.

The region E cannot be empty since $|B| + |E| = \frac{n}{3} + 2$ and $|B| \leq \frac{n}{3}$. Let p be any point in the region E. We claim that p is contained in all skylines that contain more than $\frac{2n}{3} + 1$ points from P.

Any skyline S that contains more than $\frac{2n}{3}+1$ points from P takes points from all three vertical slabs and from both horizontal slabs. Therefore S contains the entire region E and therefore the point p.

□

Theorem 7. $\frac{n^2}{9} \leq s^{\mathcal{K}}(n) \leq \frac{n^2}{8}$

Let $p \in P$ be a point as described in lemma 6 i.e., any skyline that contains more than $\frac{2n}{3}$ points from P contains p. We claim that p is contained in at least $\frac{n^2}{9}$ induced skylines.

Let p divide the plane into four quadrants as shown in figure 3. Therefore,

$$f_p^{\mathcal{K}} = |A||C| + |B||D| + |A||B|$$

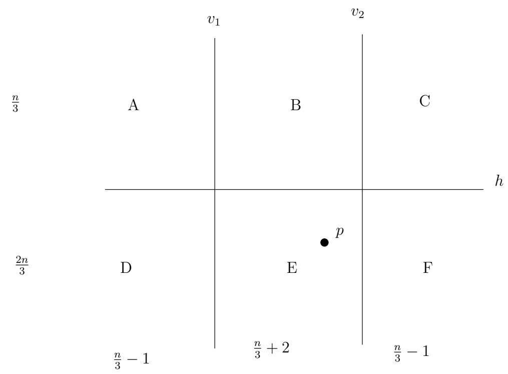


Figure 8: To prove lemma 6

If both $|A|$ and $|C|$ are $\geq \frac{n}{3}$ then the claim is true. Therefore assume this is not true. Now there are four cases. In all the cases, we fix the number of points in A and C. Note that the value of $f_p^{\mathcal{K}}$ is minimized when B has very few points than D.

1. $|A| = \frac{n}{3} - x, |C| = \frac{n}{3} - y, x \leq y$:

$$f_p^{\mathcal{K}} = \left(\frac{n}{3} - x\right)\left(\frac{n}{3} - y\right) + |B||D| + \left(\frac{n}{3} - x\right)|B|$$

Since $\frac{n}{3} \leq |A| + |D| \leq \frac{2n}{3}$ and $|C| + |D| \leq \frac{2n}{3}$,

$$y \leq |D| \leq \frac{n}{3} + x$$

Also,$|A| + |B| \geq \frac{n}{3}$ and $\frac{n}{3} \leq |B| + |C| \leq \frac{2n}{3}$. Therefore,

$$y \leq |B| \leq \frac{n}{3} + y$$

The value of $f_p^{\mathcal{K}}$ is minimized when $|B| = y$ and $|D| = \frac{n}{3} + x$. Therefore,

$$\begin{aligned} f_p^{\mathcal{K}} &= \left(\frac{n}{3} - x\right)\left(\frac{n}{3} - y\right) + y\left(\frac{n}{3} + x\right) + \left(\frac{n}{3} - x\right)y \\ &= \frac{n^2}{9} + xy + \frac{n}{3}(y - x) \\ &\geq \frac{n^2}{9} \end{aligned}$$

2. $|A| = \frac{n}{3} - x, |C| = \frac{n}{3} - y, x \geq y$:

Since $\frac{n}{3} \leq |A| + |D| \leq \frac{2n}{3}$ and $|C| + |D| \leq \frac{2n}{3}$,

$$x \leq |D| \leq \frac{n}{3} + y$$

Also,$|A| + |B| \geq \frac{n}{3}$ and $\frac{n}{3} \leq |B| + |C| \leq \frac{2n}{3}$. Therefore,

$$x \leq |B| \leq \frac{n}{3} + y$$

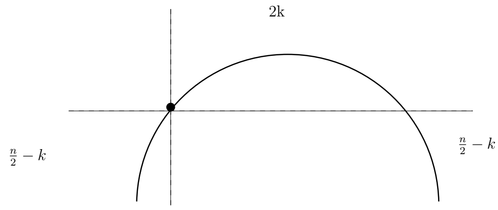


Figure 9: Upper bound construction for skylines

The value of $f_p^{\mathcal{K}}$ is minimized when $|B| = x$ and $|D| = \frac{n}{3} + y$.

$$\begin{aligned} f_p^{\mathcal{K}} &= \left(\frac{n}{3} - x\right)\left(\frac{n}{3} - y\right) + x\left(\frac{n}{3} + y\right) + x\left(\frac{n}{3} - x\right) \\ &= \frac{n^2}{9} + (x - y)\left(\frac{n}{3} + y - x\right) + y^2 \\ &\geq \frac{n^2}{9} \end{aligned}$$

3. $|A| = \frac{n}{3} - x, |C| = \frac{n}{3} + y$:

   By reasons similar to case 2,

$$x \leq |B| \leq \frac{n}{3} - y$$
$$x \leq |D| \leq \frac{n}{3} - y$$

   Therefore, the value of $f_p^{\mathcal{K}}$ is minimized when $|B| = x$ and $|D| = \frac{n}{3} - y$. Since $x \leq \frac{n}{3}$, this case now becomes exactly like one of the previous cases where two diagonally opposite quadrants have less than $\frac{n}{3}$ points.

4. $|A| = \frac{n}{3} + x, |C| = \frac{n}{3} - y$:

   Here $|B| + |D| = \frac{n}{3} + y - x$. Also,

$$y \leq |B| \leq \frac{n}{3} - y$$
$$|D| \leq \frac{n}{3} - x$$

   Therefore, the value of $f_p^{\mathcal{K}}$ is minimized when $|B| = y$ and $|D| = \frac{n}{3} - x$. Since $y \leq \frac{n}{3}$, this becomes exactly like case 1 or 2.

Therefore,

$$s^{\mathcal{K}}(n) \geq \frac{n^2}{9}$$

To show the upper bound we consider P as shown in figure 9. n points are arranged along the boundary of a semicircle. Let p be any point in P. We claim that p is contained in at most $\frac{n^2}{8}$ induced skylines.

Assume that p is the kth point from the topmost point. Therefore,

$$\begin{aligned} f_p^{\mathcal{K}} &= 2k\left(\frac{n}{2} - k\right) \\ &= nk - 2k^2 \end{aligned}$$

The value of $f_p^{\mathcal{K}}$ is maximized when $k = \frac{n}{4}$ and $f_p^{\mathcal{K}} \leq \frac{n^2}{8}$.

4 Boxes in $\mathbb{R}^d$

Let P be a set of n points in $\mathbb{R}^d$ and $\mathcal{B}$ be the set of all $\binom{n}{2}$ boxes induced by P. Let $B(a, b)$ be the box induced by $a, b \in P$ i.e, box $B(a, b)$ has a and b as diagonal points. We define $\mathcal{B}_p \subseteq \mathcal{B}$ as the set of boxes which contain a point $p \in \mathbb{R}^d$. We look at a lower bound for $w^{\mathcal{B}}(n)$.

Theorem 8. *For $d \geq 2$, $w^{\mathcal{B}}(n) \geq \frac{n^2}{2^{(2^d-1)}}$.*

Proof. We prove this by induction on the dimension d. The base case $d = 2$ is true from Theorem 1.

For $d \geq 3$, we assume that the statement is true for induced boxes in $d - 1$ dimensions. We project all the points of P orthogonally onto h, which is a $(d - 1)$ dimensional hyperplane $x_d = 0$. From the induction hypothesis, there exists a point $q = (q_1, \ldots, q_{d-1})$ in this hyperplane which is present in $\frac{n^2}{2^{2^{(d-1)}-1}}$ of the $(d - 1)$ dimensional boxes induced by the projections of P on h.

Consider the line perpendicular to the hyperplane h, which passes through q. The line l passes through those d dimensional boxes, whose projections onto h contained q. We project only these boxes onto l and look at the problem of second selection lemma for intervals ($d = 1$), where the number of points is n and the number of intervals is $\frac{n^2}{2^{2^{(d-1)}-1}}$. From lemma 14, we see that there exists a point q_d in $\frac{n^2}{2^{2^d-1}}$ intervals. This in turn gives us a point $r = (q_1, \ldots, q_d)$ which is present in the corresponding boxes in $\mathbb{R}^d$. □

Theorem 9. *For $d \geq 2$, $w^{\mathcal{B}}(n) \leq \frac{n^2}{2^{d+1}} + o(n^2)$*

Proof. Consider a set of n points which is arranged as a uniform $2^k \times 2^k \times .... \times 2^k$ d-dimensional grid, where $k = \frac{1}{d}$. [1]

Construct d hyperplanes parallel to the coordinate axes, $H = \{h_1, h_2, ..., h_d\}$ which are perpendicular to each other, each of which divides the grid into 2 halves, containing $\frac{n}{2}$ points each. Let the intersection point of these d hyperplanes be p. Now, each of the 2^d orthants defined by H contains a smaller d-dimensional uniform grid of size $\frac{n^{\frac{1}{d}}}{2} \times \frac{n^{\frac{1}{d}}}{2} \cdots \times \frac{n^{\frac{1}{d}}}{2}$. Thus, each of the orthants contain exactly $\frac{n}{2^d}$ points. The number of boxes which contain p is given by $(2^{d-1} \cdot \frac{n^2}{2^{2d}})$. The term 2^{d-1} is the number of opposite orthant pairs, whose points contribute to a box containing p. Thus, $|\mathcal{B}_p| = \frac{n^2}{2^{d+1}}$.

Consider any point $q \in \mathbb{R}^d$ (not necessarily from P), which is present inside the grid. Construct d orthogonal hyperplanes $L = \{l_1, l_2, ..., l_d\}$ parallel to H,

[1] We eliminate all the degenerate rectangles i.e. the set of all rectangles induced by two points which are along any row of the grid in each dimension. Note that for $d \geq 2$, the number of such degenerate rectangles is at most $dn^{(1+\frac{1}{d})} = o(n^2)$.

all of which intersect at q. Let r_k be the number of grid points in the k^{th} dimension between p and q (h_i and l_i).

Consider the d-dimensional uniform grids present in each of the orthants formed by L. Let us fix a dimension k, where $k \in [d]$. Consider any orthant O realized by L and let G' be the grid present in O. Let n_1 be the number of points present in the k^{th} dimension in O. This means that the diagonally opposite orthant O' to O contains $n_2 = n^{\frac{1}{d}} - n_1$ points in the k^{th} dimension. W.l.o.g, let $n_1 = \frac{n^{\frac{1}{d}}}{2} - r_k$ and thus, $n_2 = \frac{n^{\frac{1}{d}}}{2} + r_k$, where $0 \leq r_i \leq \frac{n^{\frac{1}{d}}}{2}, \forall i \in [d]$. This is true for points along any dimension $1 \leq k \leq d$, in any orthant defined by L. W.l.o.g, let G' be of size $\prod_{i=1}^{d} \left(\frac{n^{\frac{1}{d}}}{2} - r_i \right)$. O' will then have a grid of size $\prod_{i=1}^{d} \left(\frac{n^{\frac{1}{d}}}{2} + r_i \right)$. Thus, the number of induced boxes contributed to $\mathcal{B}_q$ by diagonally opposite orthants O and O', is $\prod_{i=1}^{d} (\frac{n^{\frac{1}{d}}}{2} - r_i)(\frac{n^{\frac{1}{d}}}{2} + r_i) = \prod_{i=1}^{d} (\frac{n^{\frac{2}{d}}}{4} - r_i^2)$. Since, this is true for every octant (having different combinations of$(\frac{n^{\frac{1}{d}}}{2} \pm r_i), \forall i \in [d]$), we get the same term in $\mathcal{B}_q$ for every pair of opposite orthants. The number of such orthant pairs is 2^{d-1} and thus, $|\mathcal{B}_q|$ is given by -

$$
\begin{aligned}
|\mathcal{B}_q| &= 2^{d-1} \cdot (\frac{n^{\frac{2}{d}}}{4} - r_1^2) \cdot (\frac{n^{\frac{2}{d}}}{4} - r_2^2) \ldots (\frac{n^{\frac{2}{d}}}{4} - r_d^2) \\
\implies |\mathcal{B}_q| &\leq \frac{n^2}{2^{d+1}}
\end{aligned}
$$

The point q is chosen arbitrarily and thus, any point in $\mathbb{R}^2$ is present in at most $\frac{n^2}{2^{(d+1)}}$ induced boxes. □

5 Hyperspheres in $\mathbb{R}^d$

Let P be a set of n points in $\mathbb{R}^d$ and $\mathcal{C}$ be the set of $\binom{n}{2}$ hyperspheres induced by P. Let $C(a, b)$ be the hypersphere induced by $a, b \in P$ i.e, $C(a, b)$ has a and b as diametrically opposite points.

5.1 Weak Variant for hyperspheres in $\mathbb{R}^d$

In this section, we obtain bounds for $w^{\mathcal{C}}(n)$.

5.1.1 General Point Sets

Lemma 10. $w^{\mathcal{C}}(n) \geq \frac{n^2}{2(d+1)}$

Proof. Let c be the centerpoint of P. Therefore any halfspace that contains c contains at least $\frac{n}{d+1}$ points. We claim that c is contained in at least $\frac{n^2}{2(d+1)}$ induced hyperspheres.

Let p be any point in P. Let H be the halfspace that contains c and whose outward normal is $\vec{cp}$. H contains at least $\frac{n}{d+1}$ points from P. Now, c is contained in a hypersphere induced by p and any point p_1 in H since $\angle pcp_1 > 90°$. Thus c is contained in at least $\frac{n}{d+1}$ induced hyperspheres where one of the inducing points is p. As this is true for any point in P, c is contained in $\frac{n^2}{2(d+1)}$ induced hyperspheres. $\square$

The upper bound construction is a trivial one and comes from the arrangement of P as a monotonically increasing line in $\mathbb{R}^d$. This gives us that any point $p \in \mathbb{R}^d$ is present in at most $\frac{n^2}{4}$ hyperspheres.

5.1.2 Centrally Symmetric Point Set

In this section, we prove tight bounds for a special class of point sets viz. centrally symmetric point sets. Let P be a centrally symmetric point set w.r.t origin i.e., for any point $p \in P$, $-p$ also belongs to P.

Theorem 11. $w^{\mathcal{C}}(n) = \frac{n^2}{4}$

Proof. The proof is similar to that of lemma 10.

Let o be the origin of the centrally symmetric point set P. Let p be any point in P. Let H be the halfspace that contains o and whose outward normal is $\vec{op}$. H contains $\frac{n}{2}$ points from P since for any point $p_1 \in P \backslash (H \cap P)$, $-p \in H \cap P$. By reasons similar to lemma 10, o is contained in at least $\frac{n^2}{4}$ induced hyperspheres.

To prove the upper bound, consider points arranged uniformly along a monotonically increasing line in $\mathbb{R}^d$. $\square$

5.2 Strong Variant for disks in $\mathbb{R}^2$

In this section, we obtain bounds on $s^{\mathcal{C}}(n)$ when $\mathcal{C}$ is the family of induced disks in $\mathbb{R}^2$.

5.2.1 General Point Sets

Theorem 12. $\frac{n^2}{16} \leq s^{\mathcal{C}}(n) \leq \frac{n^2}{9}$

Proof. The lower bound follows from theorem 2 since the axis-parallel rectangle induced by two points p, q are completely contained inside the disk induced by p and q.

To prove the upper bound, we use a configuration from [12]. n points are arranged as equal subsets of $\frac{n}{3}$ points, each along small circular arcs at the vertices of a triangle $\triangle ABC$. Let $A_1 = \{a_1, a_2, \cdots, a_{\frac{n}{3}}\}$ represent the points near the vertex A. Similarly, let $B_1 = \{b_1, b_2, \cdots, b_{\frac{n}{3}}\}$ represent the points near vertex B and $C_1 = \{c_1, c_2, \cdots, c_{\frac{n}{3}}\}$ represent the points near vertex C. The angles of the triangle and the length of the arcs are so selected such that the only obtuse-angled triangles are of type $\triangle a_i a_j a_k, \triangle b_i b_j b_k, \triangle c_i c_j c_k, \triangle a_i b_j b_k, \triangle b_i c_j c_k, \triangle c_i a_j a_k$ where $1 \leq i, j, k \leq \frac{n}{3}$(See section 5 in [12]).

We claim that any point $p \in P$ is contained in at most $\frac{n^2}{9}$ induced disks. W.l.o.g assume that $p \in A_1$. Also assume that p has x points of A_1 above it(i.e, away from C). The triangle with one vertex as p is obtuse when both the other two vertices are from A_1 or B_1 or one of them is from A_1 and the other is from C_1. When both the vertices are from B_1, the angle subtended at p is acute. The angles are obtuse in the following cases:

1. The other two vertices are a_i and a_j, $1 \leq i, j \leq \frac{n}{3}$ and a_i and a_j lies on either side of p in A_1.

2. The other two vertices are a_i and c_j, $1 \leq i, j \leq \frac{n}{3}$ and a_i lies above p in A_1.

Therefore,

$$\begin{aligned} f_p^{\mathcal{C}} &= x(\frac{n}{3} - x) + \frac{n}{3}(\frac{n}{3} - x) \\ &= \frac{n^2}{9} - x^2 \end{aligned}$$

The value of $f_p^{\mathcal{C}}$ is maximized when $x = 0$. Therefore, $f_p^{\mathcal{C}} \leq \frac{n^2}{9}$.

□

5.2.2 Centrally Symmetric Point sets

In this section, we prove tight bounds for centrally symmetric point sets. Let P be a centrally symmetric point set w.r.t origin.

Theorem 13. $s^{\mathcal{C}}(n) = \frac{n^2}{8}$

Proof. **Lower Bound**

Let P be a centrally symmetric point set. We claim that there exists a point $p \in P$ such that p is contained in $\frac{n^2}{8}$ disks induced by P.

We find the point $p \in P$ as follows. Let $P_1 = P$. For $i \in [1, \frac{n}{2}]$, let $a_i \in P_i$ be the point with maximum distance from the origin and let $b_i = -a_i$. The disk induced by a_i and b_i contains all the points of P_i. Otherwise, if there is a point $a_j \in P_i$ outside this disk then the distance from a_j to origin is more than the distance from a_i to origin, a contradiction. Let $P_{i+1} = P_i \setminus \{a_i, b_i\}$. Since $b_i = -a_i$, P_{i+1} is also centrally symmetric. Let $p \in P_{n/2}$. Then p has the desired property.

Let $q \in P_{j+1}$. Then we claim that q is contained in at least $\frac{j^2}{2}$ induced disks.

Let $i < j$. Clearly q is contained in $C_{a_i b_i}$ and $C_{a_j b_j}$. We claim that q is also contained in C_{ab} where $a, b \in \{a_i, a_j, b_i, b_j\}$ and C_{ab} is not $C_{a_i b_i}$ or $C_{a_j b_j}$. Assume for contradiction that this if false. Therefore $\angle a_i q a_j, \angle a_i q b_j, \angle b_i q a_j, \angle b_i q b_j$ are all acute. Consider the line segment joining a_i and q. Let h_a be the line perpendicular to this line segment and passing through q. Let H_a be the halfspace defined by h_a containing the point a_i(See figure 10). Since angles $\angle a_i q a_j$ and $\angle a_i q b_j$ are acute, both a_j and b_j belong to H_a. Now consider the line segment joining b_i and q. Define H_b as before. By similar reasoning as before, a_j and b_j belong to H_b. Therefore, both a_j and b_j belong to $H_a \cap H_b$. This contradicts the fact that $\angle a_j q b_j$ is obtuse. Therefore, at least one of the angles $\angle a_i q a_j, \angle a_i q b_j, \angle b_i q a_j, \angle b_i q b_j$ is obtuse and the disk induced by the corresponding points contains q.

Therefore, q is contained in all disks of the form $C_{a_i b_i}$ where $1 \leq i \leq j$. Also, we just proved that for any $i, k \leq j$, q is contained in at least one disk of the form C_{ab} where $a \in \{a_k, b_k\}$ and $b \in \{a_i, b_i\}$. Therefore q is contained in $\frac{j^2}{2}$ induced disks.

Since $p \in P_{n/2}$, p is contained in $\frac{n^2}{8}$ induced disks.

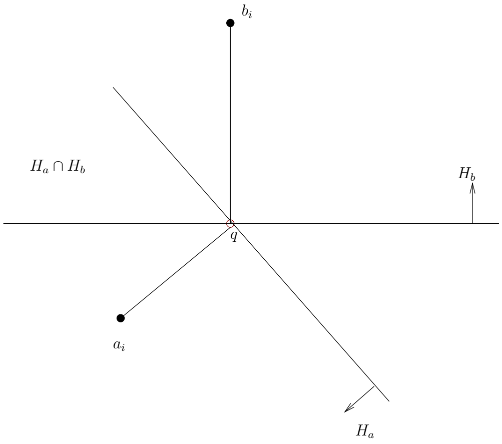


Figure 10: Lower Bound for disks

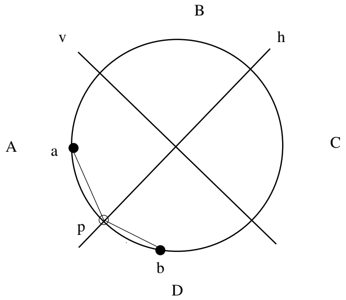


Figure 11: Upper bound for Circles

Upper Bound

Consider P as n points arranged along the boundary of a circle. P is a centrally symmetric point set. We claim that any point $p \in P$ is contained in at most $\frac{n^2}{8}$ induced disks.

Let h be a straight line connecting p and its diametrically opposite point and v be a straight line perpendicular to h. Let h and v divide the plane into four quadrants as shown in figure 11. Let $a, b \in P$. If both a and b lie in the same side of h, $\angle apb < 90$ and p is not contained in the disk induced by a and b. Therefore, assume that a and b lie on different sides of h. W.l.o.g assume that $a \in A \cup B$ and $b \in C \cup D$. Let a be the jth point from p(clockwise) and b be the kth from p(anti-clockwise), $j, k \in [1, \frac{n}{2} - 1]$. It can be clearly seen that $\angle apb \geq 90$ when $j \in [1, \frac{n}{2} - k]$. Therefore, p is contained in $1 + 2 + ... + \frac{n}{2} = \frac{n^2}{8}$ induced disks.

□

6 Second Selection lemma

In the second selection lemma, we are given an arbitrary subset $\mathcal{S} \subseteq \mathcal{R}$ of size m. We bound the maximum number of induced rectangles of $\mathcal{S}$ that can be pierced by a single point p. The main idea of our approach is an elegant double counting argument, which we first illustrate for the special case of intervals in $\mathbb{R}$.

6.1 Second selection lemma for intervals in $\mathbb{R}$

Let $P = \{x_1, x_2, ..., x_n\}$ be a set of n points in $\mathbb{R}$. For any two points $p < q$ on the real line, we call $[p, q]$ as the interval defined by the points p and q. Let C be the given set of m intervals which are induced by P, where $m \leq \binom{n}{2}$.

Let J_c denote the number of points from P present in an interval $c \in I$ and I_p denote the number of intervals in C containing the point p. Let us partition

C in such a way that, each interval with the point x_i as its left endpoint is placed in a set of intervals $X_i, \forall x_i \in P$. The intervals in X_i are ordered by their right endpoint. Let each $|X_i|$ be m_i and hence $\sum_{i=1}^{n} m_i = m$.

Lemma 14. *Let $P = \{x_1, ..., x_n\}$ be a set of n points in $\mathbb{R}$ and C be a set of m intervals induced from P. If $m = \Omega(n)$, then there exists a point $p \in P$ which is present in at least $\frac{m^2}{2n^2} + \frac{3m}{2n}$ intervals of C.*

Proof. First, let us find the lower bound for the number of points present in all the intervals in X_i. In X_i, we can see that the j^{th} interval contains at least $j+1$ points. Thus, the summation of the number of points present in the intervals of X_i is given by

$$\sum_{r \in X_i} J_r \geq 2 + 3 + ... + (m_i + 1) \geq \frac{m_i^2}{2} + \frac{3m_i}{2}$$

Each interval belongs to a unique X_i and thus, the summation of the number of points present in the intervals in C is lower bounded by summing over all x_i, the number of points present in each X_i.

$$\sum_{c \in I} J_c \geq \sum_{i=1}^{n} (\frac{m_i^2}{2} + \frac{3m_i}{2})$$
$$\geq \frac{\sum_{i=1}^{n} m_i^2}{2} + \frac{3 \cdot \sum_{i=1}^{n} m_i}{2}$$

Now, from the Cauchy-Schwarz inequality in $\mathbb{R}^n$ we have, $(\sum_{j=1}^{n} m_j^2) \geq \frac{m^2}{n}$.
Thus, $\sum_{c \in I} J_c \geq \frac{m^2}{2n} + \frac{3m}{2}$

Now, the count we are achieving by summing over the number of points present in an interval J_c, can also be gotten through by summing over the number of intervals containing a point I_p.

$$\sum_{c \in I} J_c = \sum_{p \in P} I_p$$
$$\implies \sum_{p \in P} I_p \geq \frac{m^2}{2n} + \frac{3m}{2}$$

By the pigeonhole principle, there exists a point $p \in P$ present in at least $\frac{m^2}{2n^2} + \frac{3m}{2n}$ intervals. □

Lemma 15. *There exists a point set P of size n and a set of induced intervals C of size $m \leq n^2(\sqrt{2} - 1) - \frac{n}{\sqrt{2}}$, such that any point in P is present in at most $\frac{m^2}{n^2} + \frac{3m}{\sqrt{2}n}$ intervals in C.*

Proof. Let $P = \{x_1, x_2, \dots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$. Let m be a multiple of n and let $m_i = \frac{\sqrt{2}m}{n}$. Let the induced intervals from C be of the form $[x_i, x_{i+1}], [x_i, x_{i+2}], ..., [x_i, x_{i+k}], \forall x_i \in P$, where $k = min(\frac{\sqrt{2}m}{n}, n - i)$. We now have, $|C| = (n - \frac{\sqrt{2}m}{n}) \cdot \frac{\sqrt{2}m}{n} + ((\frac{\sqrt{2}m}{n} - 1) + \cdots + 1) = \sqrt{2}m - (\frac{m^2}{n^2} + \frac{m}{\sqrt{2}n})$. It is easy to see that $|C| \geq m$, when $m \leq n^2(\sqrt{2} - 1) - \frac{n}{\sqrt{2}}$.

Let $B \subset P$ be the set of points, which exclude the first and the last $\frac{\sqrt{2}m}{n}$ points from P. Consider any point $x_p \in B$. Let us count the number of intervals containing x_p i.e I_{x_p}.

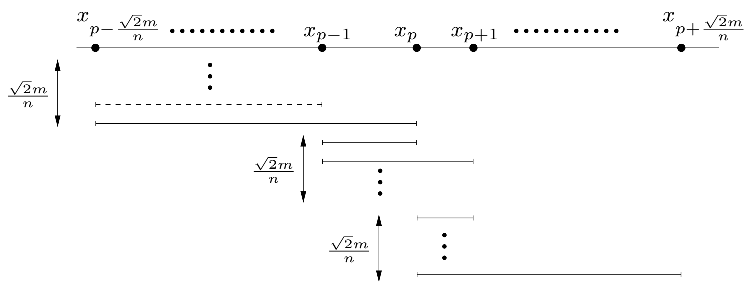


Figure 12: The intervals which contain x_p (Bold intervals)

From the figure 12, it can be seen that there is only one interval from $I_{x_{(p-\frac{\sqrt{2}m}{n})}}$ which contains x_p. This count of intervals containing x_p increases by 1 for each consecutive point after $x_{p-\frac{\sqrt{2}m}{n}}$, until we reach x_{p-1} and x_p, both of which have $\frac{\sqrt{2}m}{n}$ intervals containing x_p.
Thus, we have

$$
\begin{aligned}
I_{x_p} &= 1 + 2 + ... + \frac{\sqrt{2}m}{n} + \frac{\sqrt{2}m}{n} \\
&= \frac{m^2}{n^2} + \frac{3m}{\sqrt{2}n}
\end{aligned}
$$

From our construction of C, it can be seen that any point $q \in P - B$ will be involved in lesser number of intervals and thus, $|I_q| < \frac{m^2}{n^2}$. The bounds are tight upto a multiplicative constant. □

6.2 Second selection lemma for Axis-Parallel Rectangles in $\mathbb{R}^2$

Let P be a set of n points in $\mathbb{R}^2$. Let $\mathcal{S} \subseteq \mathcal{R}$ be any set of m induced axis-parallel rectangles. In the second selection lemma, we bound the maximum number of induced rectangles of $\mathcal{S}$ that can be pierced by a single point p. The main idea of our approach is an elegant double counting argument.

Let $R(p, q)$ denote the rectangle induced by the points p and q. $\mathcal{S}$ is partitioned into sets X_i as follows : any rectangle $R(x_i, u) \in \mathcal{S}$ where $x_i, u \in P$, is added to the partition X_i if u is higher than x_i. Let $P_i = \{u | R(x_i, u) \in X_i\}$. Let $|P_i| = |X_i| = m_i$. Any rectangle $R(x_i, u) \in X_i$ is placed in one of two sub-partitions, X_i' or X_i'', depending on whether u is to the right or left of x_i. Let $|X_i'| = m_i'$ and $|X_i''| = m_i''$. Similarly, we partition P_i into P_i' and P_i''. Let $\sum_{i=1}^{n} m_i' = m'$ and $\sum_{i=1}^{n} m_i'' = m''$. The rectangles in X_i' (or X_i'') and the points in P_i' (or P_i'') are ordered by decreasing y-coordinate.

We construct a grid out of P by drawing horizontal and vertical lines through each point in P. Let the resulting set of grid points be G ($P \subset G$), where $|G| = n^2$. We use the grid points in G as the candidate set of points for the second selection lemma.

Let J_r be the number of grid points in G present in any rectangle $r \in \mathcal{S}$. W.l.o.g consider the set of rectangles present in X_i'. We obtain a lower bound on $\sum_{r \in X_i'} J_r$.

Lemma 16. $\displaystyle\sum_{r \in X_i'} J_r \geq \frac{(m_i')^3}{6}$.

Proof. Let $c = \sum_{r \in X_i'} J_r$. We prove the lemma by induction on the size of m_i'. For the base case, let $m_i' = 2$. There are only two ways in which the point set can be arranged, as shown in figure 13(a). It can be seen that the statement is true for the base case.

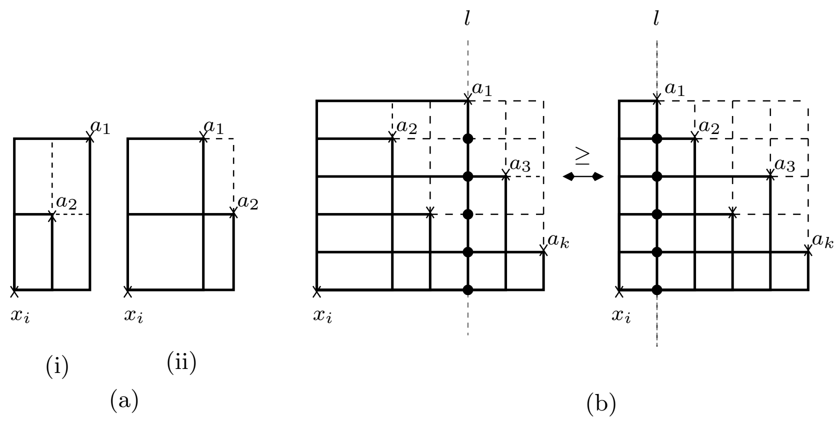


Figure 13: The dotted lines represent the grid lines and the solid lines represent the rectangle edges. (a) Base cases. (b) Inductive case - the case when a_1 is not the leftmost point in P_i'.

For the inductive case, assume that the statement is true for $m_i' = k-1$ and let $m_i' = k$. Let $P_i' = \{a_1, a_2, ..., a_k\}$. Let a_1 be the topmost point in P_i' as seen in figure 13(b) and l be the vertical line passing through a_1. We have 2 cases :

Case 1 : If a_1 is the leftmost point in P_i', then we remove a_1 from P_i' and $R(x_i, a_1)$ from X_i'. By the induction hypothesis, the lemma is true for the remaining $k-1$ points. On adding a_1 back, we see that the line l contributes k grid points to the next rectangle in X_i', $R(x_i, a_2)$. This contribution of grid points by l becomes $k-1$ for the next rectangle $R(x_i, a_3)$ and decreases by one as we move through the ordered set X_i' and it is two for $R(x_i, a_k)$. Thus, the total number of points contributed by l to c is given by $\frac{k(k+1)}{2} - 1$. The rectangle $R(x_i, a_1)$ also contributes $2k+2$ to c. Thus, $c \geq \frac{(k-1)^3}{6} + \frac{k(k+1)}{2} + (2k+1) \geq \frac{k^3}{6}$. Thus, the statement is true for $m_i' = k$.

Case 2 : If a_1 is not the leftmost point, then we claim that c does not increase when we make a_1 as the leftmost point by moving line l to the left. To see this, refer figure 13(b) where the grid points on l are shown as solid circles. Let j be the number of points from P_i' present to the left of l. When we make the point a_1 as the leftmost by moving l to the left, we see that

- The rectangles induced by x_i and the points to the left of l have an increase in the number of grid points, which is contributed by l. Thus, c increases by $t \leq k + (k-1) + ... + (k-j+1) = \frac{j(2k+1-j)}{2}$.
- $R(x_i, a_1)$ loses $d = (j+2)(k+1) - 2(k+1) = j(k+1)$ points. Thus, c decreases by d.
- The number of grid points in the rectangles induced by x_i and the points to the right of l remains the same.

By a simple calculation we can see that $d \geq t$. Thus, when a_1 is moved to the left, c does not increase. As a_1 is now the leftmost point, we can apply case 1 and show that the lemma is true for $m_i' = k$. □

Theorem 17. *Let P be a point set of size n in $\mathbb{R}^2$ and let $\mathcal{S}$ be a set of induced rectangles of size m. If $m = \Omega(n^{\frac{4}{3}})$, then there exists a point $p \in G$ which is present in at least $\frac{m^3}{24n^4}$ rectangles of $\mathcal{S}$.*

Proof. The summation of the number of grid points present in the rectangles in X_i is given by $\sum_{r \in X_i} J_r = \sum_{r \in X_i'} J_r + \sum_{r \in X_i''} J_r$. Using the lower bound from lemma 16 we have, $\sum_{r \in X_i} J_r \geq \frac{(m_i')^3 + (m_i'')^3}{6}$.

Since $\mathcal{S}$ is partitioned into the sets X_i, the summation of the number of grid points present in the rectangles in $\mathcal{S}$ is given by

$$\sum_{r \in \mathcal{S}} J_r = \sum_{i=1}^{n} \sum_{r \in X_i} J_r \geq \left(\sum_{i=1}^{n} (m_i')^3 + \sum_{i=1}^{n} (m_i'')^3 \right) / 6$$

Using Hölder's inequality in $\mathbb{R}^n$ (generalization of the Cauchy-Schwartz inequality), we have $\sum_{i=1}^{n} (m_i')^3 \geq \frac{(m')^3}{n^2}$. Thus, we get $\sum_{r \in \mathcal{S}} J_r \geq \frac{(m')^3 + (m'')^3}{6n^2}$. This sum is minimized when $m' = m'' = \frac{m}{2}$ and thus, $\sum_{r \in \mathcal{S}} J_r \geq \frac{m^3}{24n^2}$.

Let I_g be the number of rectangles of $\mathcal{S}$ containing the grid point $g \in G$. Now, by double counting, we have

$$\sum_{g \in G} I_g = \sum_{r \in \mathcal{S}} J_r \implies \sum_{g \in G} I_g \geq \frac{m^3}{24n^2}$$

By pigeonhole principle, there exists a grid point $p \in G$ which is present in at least $\frac{m^3}{24n^4}$ rectangles in $\mathcal{S}$. □

6.3 Second selection lemma for other objects in $\mathbb{R}^2$

In this section, we look at the second selection lemma for objects like skylines and downward facing equilateral triangles.

Smorodinsky and Sharir [20] proved tight bounds for the second selection lemma for disks. They used the planarity of the Delaunay graph (w.r.t circles) to prove that there exists a point $p \in P$ which is present in at least $\Omega(\frac{m^2}{n^2})$ disks of D. It is not hard to see that this result applies for all objects whose Delaunay graph is planar.

6.3.1 Skylines

Let $\mathcal{K}' \subseteq \mathcal{K}$ be a set of m skylines induced by P. It can be easily seen that the Delaunay graph w.r.t skylines is planar. We can directly use the result in [20] to get upper and lower bounds on the second selection lemma for induced skylines.

Lemma 18. *There exists a point $p \in P$, which is present in $\Omega(\frac{m^2}{n^2})$ skylines induced by P. This bound is asymptotically tight.*

6.3.2 Downward facing equilateral triangles

Let $\mathcal{T}$ be the set of all downward facing equilateral triangles or down-triangles induced by P. Such a triangle is induced by two points where the side parallel to the x-axis passes through one of the points and the corner opposite to this side lies below it. The other inducing point is present on one of the other 2 sides. Let $\mathcal{T}' \subseteq \mathcal{T}$ be a set of m induced down-triangles. [5] proved that the Delaunay graph w.r.t to down-triangles is planar. Thus, we can apply the result in [20] directly to get upper and lower bounds.

Lemma 19. *There exists a point $P \in P$, which is present in $\Omega(\frac{m^2}{n^2})$ down-triangles induced by P. This bound is asymptotically tight.*

7 Conclusion

In this paper, we have studied selection lemma type questions for various geometric objects. We have proved exact results for both the strong and weak variants of the first selection lemma for axis-parallel rectangles and special subclasses like quadrants and slabs. For the weak variant of the first selection lemma for axis-parallel boxes in $\mathbb{R}^d$ though, there is a wide gap between our lower bounds ($\frac{n^2}{2^{(2^d-1)}}$) and our upper bounds ($\frac{n^2}{2^{d+1}}$), which needs to be tightened. We have shown non trivial bounds for the weak variant of first selection lemma for induced hyperspheres. Finding the exact constant is an interesting open problem. Another open problem is to find non-trivial bounds for the strong variant of first selection lemma for boxes and hyperspheres in higher dimensions.

For the second selection lemma for axis-parallel rectangles, we have proved a lower bound of $\frac{m^3}{24n^4}$ which is a better bound than [20], when $m = \Omega(\frac{n^2}{\log^2 n})$. An interesting open problem, as mentioned in [20], is to tighten the polylogarithmic gap between these lower and upper bounds.

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