Hitting and Piercing Rectangles Induced by a Point Set

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June 21, 2013

Induced Geometric Objects

P - set of n points in $\mathbb{R}^2$ in general position.
$\mathcal{R}$ - Set of **all** distinct geometric objects of a particular class induced(spanned) by P.
For example, let $\mathcal{R}$ be the set of **all** the $\binom{n}{2}$ axis-parallel rectangles induced by a distinct pair of points in P.

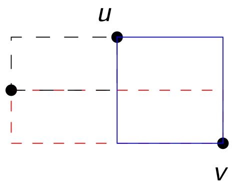

FIGURE: Set of all axis-parallel Rectangles induced by P

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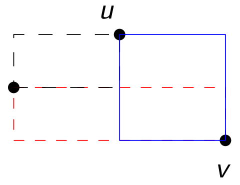

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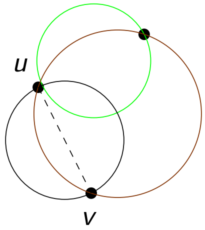

FIGURE: Set of all diametrical Disks induced by P

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2. What is the minimum set of points needed to hit all the objects in $\mathcal{R}$? (Hitting Set)

First Selection Lemma (FSL)

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2. For induced simplices in $\mathbb{R}^d$, Bárány (1982) showed that there exists a point $p \in \mathbb{R}^d$ contained in at least $c_d \cdot \binom{n}{d+1}$ simplices induced by P. Result used in the construction of weak ϵ-nets for convex objects (Matousek 2002).

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3. FSL type results have not been explored for other classes of induced objects like axis-parallel rectangles, disks etc.
4. Strong first selection lemma ($p \in P$).

Second Selection Lemma (SSL)

1. Generalization of the first selection lemma, which considers an m-sized arbitrary subset $\mathcal{S} \subseteq \mathcal{R}$ and shows that there exists a point which is contained in $f(m, n)$ objects of $\mathcal{S}$.

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5. Smorodinsky et al.(2004) gave an alternate proof of the same bounds and also gave an upper bound of $O(\frac{m^2}{n^2 \log(\frac{n^2}{m})})$.

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3. Combinatorial Bounds studied for induced disks, axis-parallel rectangles and triangles.

Our Results

1. Selection Lemmas
   1. For the first selection lemma for axis-parallel rectangles, we show a tight bound of $\frac{n^2}{8}$.
   2. For the strong first selection lemma for axis-parallel rectangles, we show a tight bound of $\frac{n^2}{16}$.
   3. (Second selection lemma) We show that $f(m,n) = \Omega(\frac{m^3}{n^4})$ for axis-parallel rectangles. Improvement over the previous bound in Smorodinsky et al.(2004), when $m = \Omega(\frac{n^2}{\log^2 n})$.
2. Hitting set for induced objects
   1. The hitting set problem for all induced lines is NP-complete.
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Some notation -

1. P - Pointset of size n in $\mathbb{R}^2$ in general position.
2. $R(u,v)$ - axis-parallel rectangle induced by u and v where $u,v \in P$.
3. $\mathcal{R}$ - set of all $R(u,v)$ for all $u,v \in P$.
4. $R_p \subset \mathcal{R}$ - set of axis-parallel rectangles that contain $p \in \mathbb{R}^2$.

FSL for Axis-Parallel Rectangles (Weak)

Theorem

Let $f(n) = \min_{P,|P|=n} (\max_{p \in \mathbb{R}^2} |R_p|)$.
There exists a point p *in* $\mathbb{R}^2$ *(not necessarily belonging to* P*), which is present in at least* $\frac{n^2}{8}$ *axis-parallel rectangles induced by* P *i.e* $f(n) \geq \frac{n^2}{8}$.
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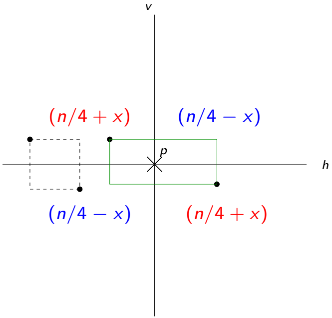

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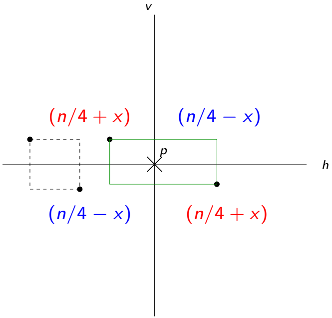

Horizontal line h and vertical line v, each of which bisects the pointset.

$|R_p| = (\frac{n}{4} - x)^2 + (\frac{n}{4} + x)^2$

$\implies |R_p| = \frac{n^2}{8} + 2x^2$

Thus, $|R_p| \geq \frac{n^2}{8}$.

SSL for Axis-Parallel Rectangles

The problem - Let $\mathcal{S} \subseteq \mathcal{R}, |\mathcal{S}| = m$. We bound the maximum number of rectangles in $\mathcal{S}$ that can be pierced by a single point $p \in \mathbb{R}^2$.

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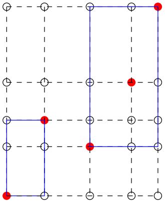

Construct a grid out of P. Let the grid points be G ($P \subset G$), where $|G| = n^2$. G is the candidate set of points for the second selection lemma.

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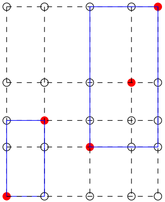

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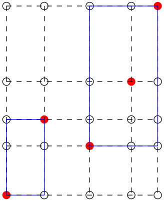
Theorem

If $m = \Omega(n^{\frac{4}{3}})$, there exists a point $p \in G$ which is present in at least $\frac{m^3}{24n^4}$ rectangles of $\mathcal{S}$.

Sketch of the proof



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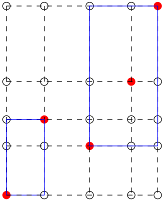

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- Same as counting the number of rectangles of $\mathcal{S}$ pierced by a grid point, summed over all grid points.
- By pigeonhole principle, we find a lower bound on the rectangles of $\mathcal{S}$ pierced by some grid point.

Number of grid points in X_i' - Lower Bound

Some notations used in the proof -

1. The rectangle $R(x_i, u) \in \mathcal{S}$ where $x_i, u \in P$ is added to the partition X_i, if u is higher than x_i (similarly P_i). Further partitioned into X_i' and X_i'' (right and left).
2. Let $|X_i'| = |P_i'| = m_i'$.
3. Let J_r be the number of grid points in G, present in any rectangle $r \in \mathcal{S}$.

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Let $c = \sum_{r \in X_i'} J_r$. *Then* $c \geq \frac{(m_i')^3}{6}$.

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Proof is by induction on m_i'.

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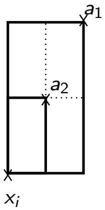

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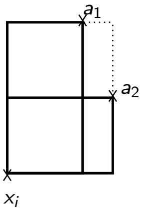

(ii)

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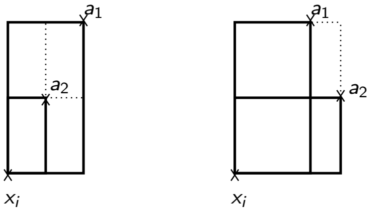
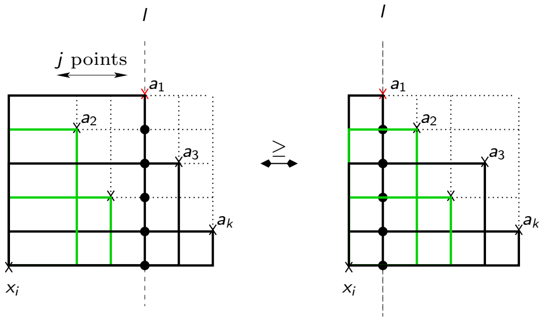


(i) (ii)

- Assume that the statement is true for $m'_i = k - 1$ and let $m'_i = k$.
- We prove that the lower bound is achieved when P'_i is monotonically decreasing i.e. any other configuration of P'_i gives a higher count for c.

Inductive Hypothesis - Case 1

Case 1 : a_1 is not the leftmost point.



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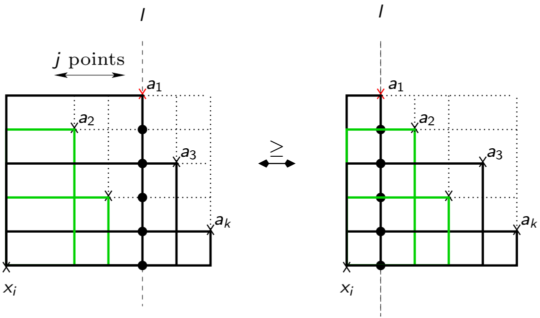

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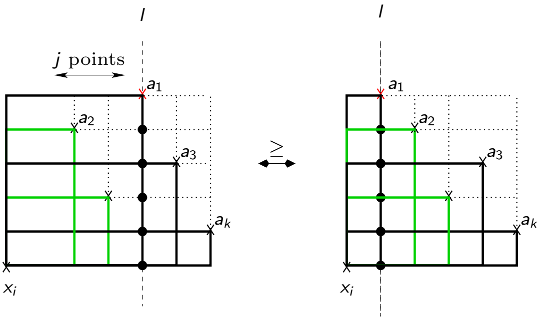

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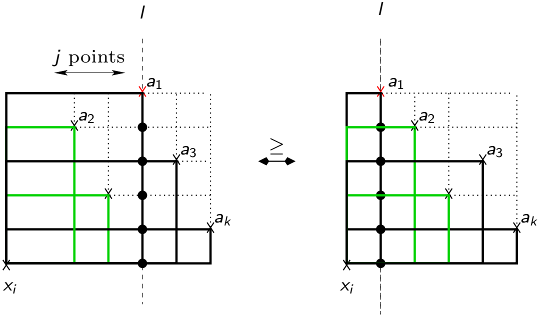

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  - The increase in c is $\leq k + (k-1) + \cdots + (k-j+1)$.
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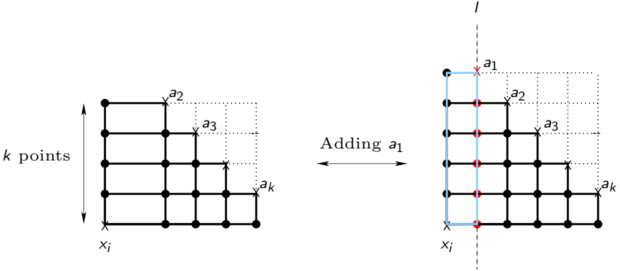
- Thus, we see that c does not increase.

Inductive Hypothesis - Case 2

Case 2 : a_1 is the leftmost point.



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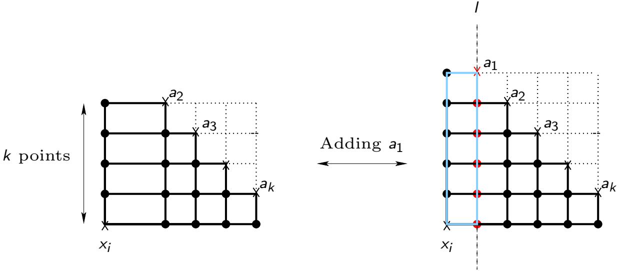

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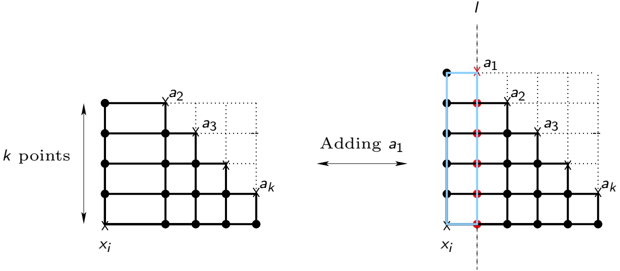

- Remove a_1 from P'_i and apply the induction hypothesis to the remaining $k-1$ points.
- The line l contributes $\frac{k(k+1)}{2} - 1$ to $\sum_{r \in X'_i} J_r$.
- $R(x_i, a_1)$ contributes $2k+2$.
- By summing all these quantities, we see that the induction hypothesis is true.

Proof of Theorem 2

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If $m = \Omega(n^{\frac{4}{3}})$, *there exists a point* $p \in G$ *which is present in at least* $\frac{m^3}{24n^4}$ *rectangles of* $\mathcal{S}$.

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Sketch of the proof -

- $\sum_{r \in \mathcal{S}} J_r = \sum_{i=1}^{n} (\sum_{r \in X_i} J_r) \geq \frac{m^3}{24n^2}$ (Lemma 3 and Hölder's inequality).
- I_g - the number of rectangles of $\mathcal{S}$ containing the grid point $g \in G$.
- $\sum_{g \in G} I_g = \sum_{r \in \mathcal{S}} J_r$

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- We use an averaging argument(n^2 grid points) and prove the theorem.

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- First selection lemma for other induced objects like disks etc.
- Can the hitting set for the set of all induced objects (disks, axis-parallel rectangles etc.), be computed in polynomial time ?