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Directed Graphs and Cannibals

Stranger in car: "How do I get to the corner of Graham Street and Harary Avenue?"

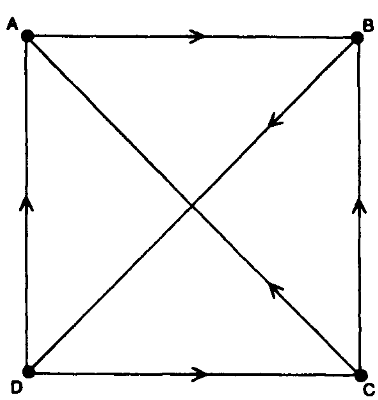
Native on sidewalk: "You can't get there from here."

In graph theory a graph is defined as any set of points joined by lines, and a simple graph is defined as one that has no loops (lines that join a point to itself) and no parallel lines (two or more lines joining the same pair of points). If an arrowhead is added to each line of a graph, giving each line a direction that orders its end points, the graph becomes a directed graph, or digraph for short. Directed lines are called arcs. Digraphs are the subject here, and the old joke quoted above is appropriate because on some digraphs it is actually impossible to get from one specified point to another.

A digraph is called complete if every pair of points is joined by an arc. For example, a complete digraph for four points is shown in Figure 39 (left). The figure at the right is the adjacency matrix of the digraph, which is constructed as follows. Think of the digraph as a map of one-way streets. Starting at point A, it is possible to go directly only to point *B*, a fact that is indicated in the top row of the matrix (the row corresponding to A) by putting a 1 in the column corresponding to *B* and a 0 in all the other columns. The remaining rows of the adjacency matrix are determined in the same way, so that the matrix is combinatorially equivalent to the digraph. It follows that given the adjacency matrix it is easy to construct the digraph.

Other important properties of digraphs can be exhibited in other kinds of matrixes. For example, in a distance matrix each cell gives the smallest number of lines that form what is called a directed path from one point to another, that is, a path that conforms to the arrowheads on the graph and does not visit any point more than once. Similarly, the cells of a detour matrix give the number of lines in the longest directed path between each pair of points. And a reachability matrix

Figure 39



	A	B	C	D
A	0	1	0	0
B	0	0	0	1
C	1	1	0	0
D	1	0	1	0

indicates (with 0s and 1s) whether a given point can be reached from another point by a directed path of any length. If every point is reachable from every other point, the digraph is said to be strongly connected. Otherwise there will be one or more pairs of points for which "you can't get there from here."

The following theorem is one of the most fundamental and surprising results about complete digraphs: No matter how the arrowheads are placed on a complete digraph, there will always be a directed path that visits each point just once. Such a path is called a Hamiltonian path after the Irish mathematician William Rowan Hamilton. Hamilton marketed a puzzle game based on a graph equivalent to the skeleton of a dodecahedron in which one task was to find all the paths that visit each point just once and return to the starting point. A cyclic path of this type is called a Hamiltonian circuit. (Hamilton's game is discussed in Chapter 6 of my *Scientific American Book of Mathematical Puzzles & Diversions*.)

The complete-digraph theorem does not guarantee that there will be a Hamiltonian circuit on every complete digraph, but it does ensure that there will be at least one Hamiltonian path. More surprisingly, it turns out that there is always an odd number of such paths. For example, on the complete digraph in Figure 40 there are five Hamiltonian paths: *ABDC*, *BDCA*, *CABD*, *CBDA*, and *DCAB*. All but one of them (*CBDA*) can be extended to a Hamiltonian circuit.

The theorem can be expressed in other ways, depending on the interpretation given the graphs. For example, complete digraphs are often called tournament graphs because they model the results of the kind of round-robin tournaments in which each player plays every other player once. If *A* beats *B*, a line goes from *A* to *B*. The theorem guarantees that whatever the outcome of a tournament is all players can be ranked in a column so that each player has defeated the player immediately below him. (It is assumed here that, as in tennis, no game can end

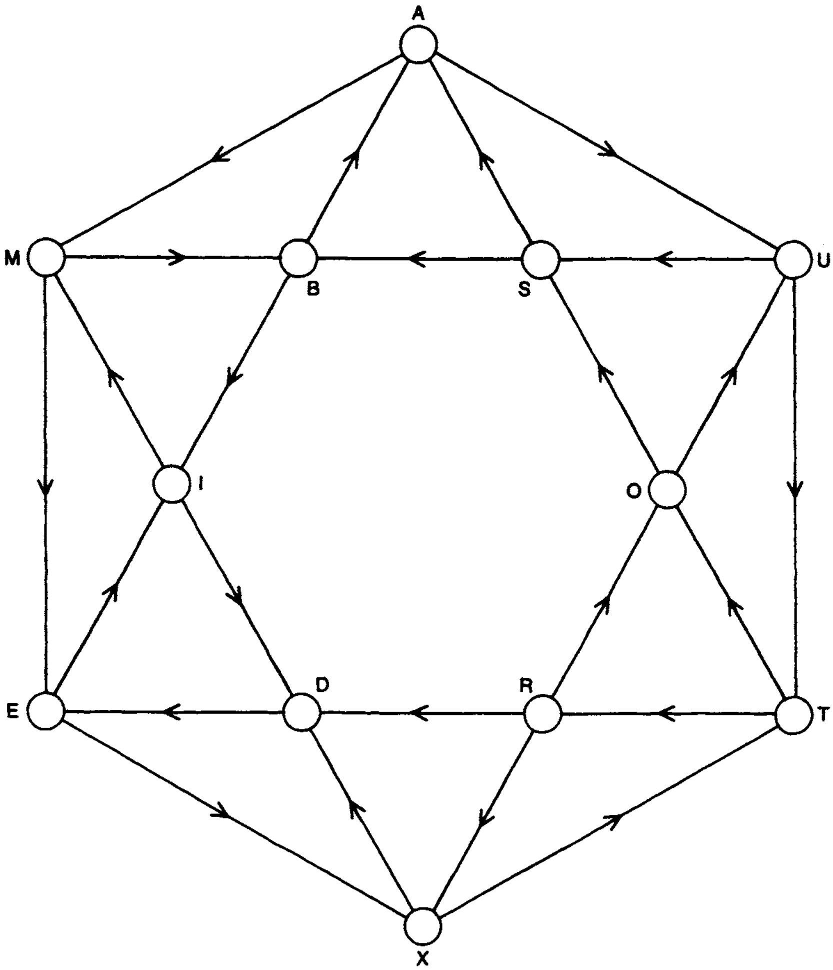


Figure 40

in a draw. If a game did allow draws, they would be represented by undirected lines and the graph would be called a mixed graph. Mixed graphs can always be converted into digraphs by replacing each undirected line with a pair of directed parallel lines going in opposite directions.)

Tournament graphs can be applied to represent many situations other than tournaments. Biologists have used the graphs to diagram the pecking order of a flock of chickens or, more generally, to diagram the structure that any other kind of pairwise dominance relation imposes on a population of animals. Social scientists have used the graphs for modeling dominance relations among people or groups of people. Tournament graphs provide a convenient means of modeling a person's pairwise preferences for any set of choices, such as brands of coffee or candidates in an election. In all these cases the theorem guarantees that the animals, people, or objects in question can always be ordered in a linear chain by means of the one-way relation.

The theorem is tricky to prove, but to convince yourself of its validity try labeling a complete graph of *n* points so that no Hamiltonian path is created. The impossibility of the task suggested the following pencil-and-paper game to the mathematician John Horton Conway. Two players take turns adding an arrowhead to any undirected line of a complete graph, and the first player to complete a Hamiltonian path loses. The theorem ensures that the game cannot be a draw. Conway finds the play is not interesting unless there are seven or more points in the graph.

The digraph in Figure 40 appeared as a puzzle in the October 1961 issue of the Cambridge mathematical annual *Eureka*. Although it is not a complete digraph, it has been cleverly labeled with arrowheads so that it has only one Hamiltonian circuit. Think of the graph as a map of one-way streets. You want to start at A and drive along the network, visiting each intersection just once before returning to A. How can it be done? (Hint: The circuit can be traced by a pencil held in either hand.)

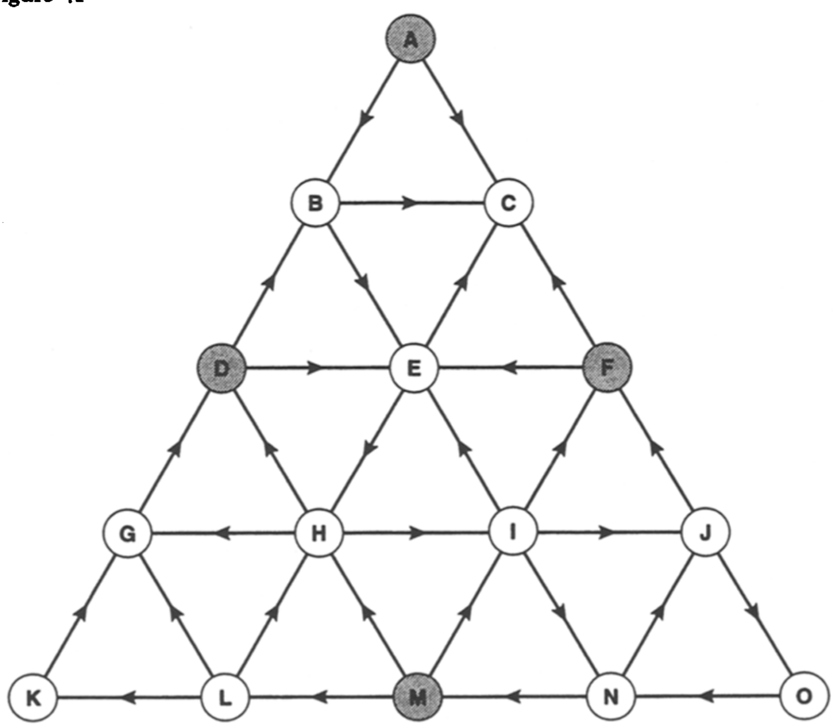
Digraphs can provide puzzles or be applied as tools for solving puzzles in innumerable ways. For example, the graphs serve to model the ways a flexagon flexes, and they are valuable in solving moving-counter and sliding-block puzzles and chess-tour problems. Probability questions involving Markov chains often yield readily to a digraph analysis, and winning strategies for two-person games in which each move alters the state of the game are frequently found by exploring a digraph of all possible plays. In principle even the game of chess could be "solved" by examining its digraph, but the graph would be so enormous and so complex that it will probably never be drawn.

Digraphs are extremely valuable in the field of operations research, where they can be applied to solve complicated scheduling problems. Consider a manufacturing process in which a certain set of operations must be performed. If each operation requires a fixed amount of time to perform and certain operations must be completed before others can be started, an optimum schedule for the operations can be devised by constructing a graph in which each operation is represented by a point and each point is labeled with a number that represents the time needed for completing the operation. The sequences in which certain operations must be done are indicated by arrowheads on the lines. To determine an optimum schedule the digraph is searched, with a computer if necessary, for a "critical path" that completes the process in a minimum amount of time. Complicated transportation problems can be handled the same way. For example, each line in a digraph can represent a road and can be labeled with the cost of transporting a particular product on it. Clever algorithms can then be applied to find a directed path that minimizes the total cost of shipping the product from one place to another.

Digraphs also serve as playing boards for some unusual board games. Aviezri S. Fraenkel, a mathematician at the Weizmann Institute of Science in Israel, has been the most creative along these lines. (For a good introduction to a class of digraph games Fraenkel calls annihilation games, see "Three Annihilation Games," a paper Fraenkel wrote with Uzi Tassi and Yaacov Yesha for *Mathematics Magazine*, Vol. 51, No. 1, pages 13–17; January 1978.) In 1976 the excellent game Arrows, which Fraenkel developed with Roger B. Eggleton of Northern Illinois University, was marketed in Israel by Or Da Industries and distributed in the U.S. by Leisure Learning Products of Greenwich, CT.

Traffic Jam, another Fraenkel game, is played on the directed graph in Figure 41. A coin is placed on each of four spots: A, *D*, *F*, and *M*. Players take turns moving any one of the coins along one of the lines of the graph to an adjacent spot as is indicated by the arrowheads on the graph. A coin can be moved to any adjacent spot whether or not the spot is occupied, and each spot can hold any number of coins. Note that all the arrowheads at C point inward. Graph theorists call such a point a sink. Conversely, a point from which all the arrowheads point outward is called a source. (If the graph models a pecking order, the sink is the chicken all the other chickens peck and the source is the chicken that pecks all the others.) In this case there is just one sink and

Figure 41



one source. (A complete digraph can never have more than one sink or more than one source. Do you see why?)

When all four coins are on sink C, the person whose turn it is to move has nowhere to go and loses the game. In Conway's book *On Numbers and Games* (Academic Press, 1976) he proves that the first player can always win if and only if his first move is from M to L. Otherwise the opponent can force a win or draw. (It is assumed that both players make their best moves.) With the powerful game theory that Conway has developed it is possible to completely analyze any game of this type, with any starting pattern of counters.

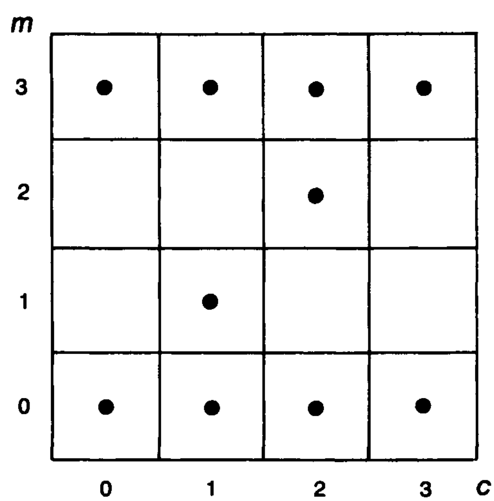
An ancient and fascinating class of puzzles that are best analyzed by digraphs are those known as river-crossing problems. Consider a classic puzzle that turned up in the title of Mary McCarthy's novel *Cannibals and Missionaries*. In the simplest version of this problem three missionaries and three cannibals on the right bank of a river want to get to the left bank by means of a rowboat that can hold no more than two passengers at a time. If the cannibals outnumber the missionaries on either bank, the missionaries will be killed and eaten. Can all six get safely across? If they can, how is it done with the fewest crossings? (I shall not enter here into the current lively debate about whether cannibalism ever actually prevailed in a culture.)

Benjamin L. Schwartz, in an article titled "An Analytic Method for the 'Difficult Crossing' Puzzles" (*Mathematics Magazine*, Vol. 34, No. 4, pages 187–193; March–April 1961), explained how to solve such problems by means of digraphs, but his method deals not directly with the digraphs but rather with their adjacency matrixes. I shall describe here a comparable procedure using the digraphs themselves that was first explained by Robert Fraley, Kenneth L. Cooke and Peter Detrick in their article "Graphical Solution of Difficult Crossing Puzzles" (*Mathematics Magazine*, Vol. 39, No. 3, pages 151–157; May 1966). The paper has been reprinted with additions as Chapter 7 of *Algorithms*, *Graphs and Computers* by Cooke, Richard E. Bellman and Jo Ann Lockett (Academic Press, 1970). The following discussion is based on that chapter.

Let m stand for the number of missionaries and c for the number of cannibals, and consider all possible states on the right bank. (It is not necessary to consider states on the left bank as well because any state on the right bank fully determines the state on the left one.) Since m can be equal to 0, 1, 2, or 3, and the same is true for c, there are 4×4, or 16, possible states, which are conveniently represented by the matrix in Figure 42. Six of these states are not acceptable, however, because the cannibals outnumber the missionaries on one of the banks. The ten acceptable states that remain are marked by placing a point inside each of the ten corresponding cells of the matrix.

The next step is to connect these points by lines that show all possible transitions between acceptable states by the transfer of one or two persons to the other side of the river. The result is the undirected graph in Figure 43. This graph is then transformed into a mixed graph by adding arrowheads to show the direction of each transition. The

Figure 42



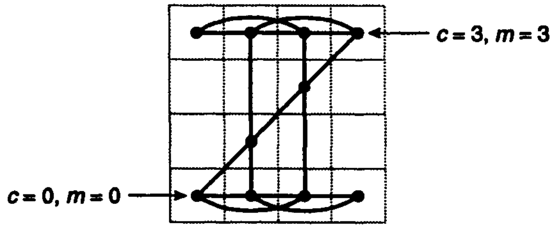


Figure 43

transformation of the undirected graph to a mixed graph must be carried out in accordance with two rules:

1. The object is to create a directed "walk" that will start at the point at the upper right ($c = 3$, $m = 3$) and end at the point at the lower left ($c = 0$, $m = 0$), so that all the cannibals and missionaries end up on the left bank. (This route is called a walk rather than a path because by definition a path cannot visit the same point more than once.)

2. The directed walk must alternate movements down or to the left with movements up or to the right, because each step down or to the left corresponds to a trip from the right bank to the left bank, whereas each step up or to the right corresponds to a trip in the opposite direction.

With both of these rules in mind it takes only a short time to discover that there are just four walks that solve the puzzle. Their digraphs are shown in Figure 44. Each walk completes the transfer in eleven moves. Note that the third through ninth steps are the same in all four walks. The four variants arise because there are two ways to make the first two steps and two symmetrical counterparts for the last two steps.

If the problem is altered to deal with transporting four cannibals and four missionaries (and all other conditions remain the same), the digraph technique can be applied to show there is no solution. Sup-

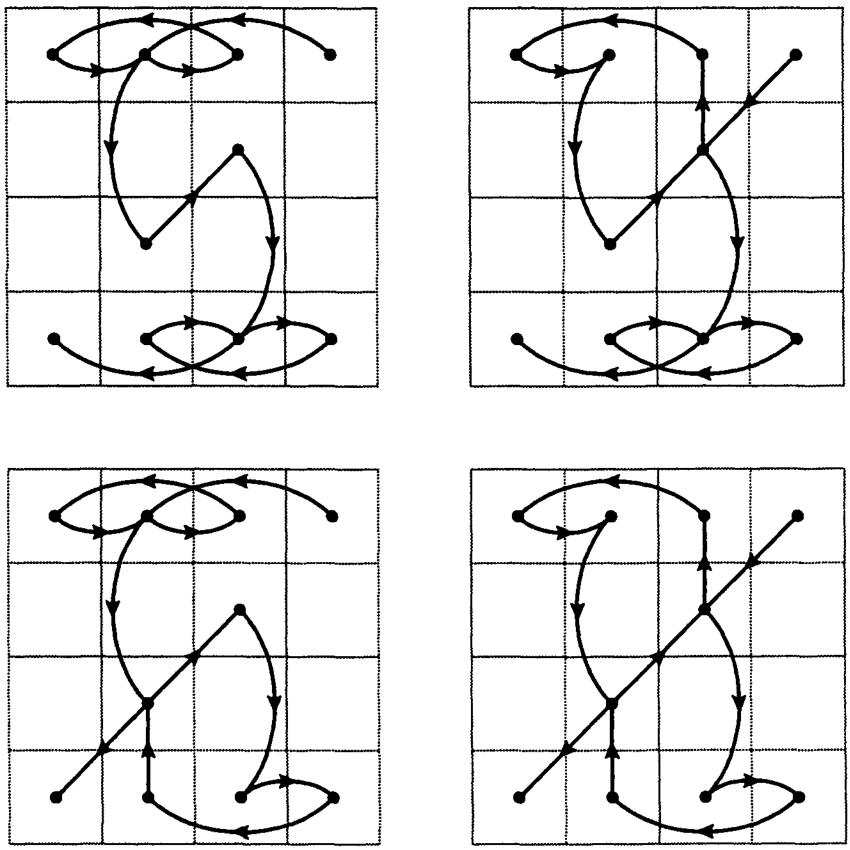
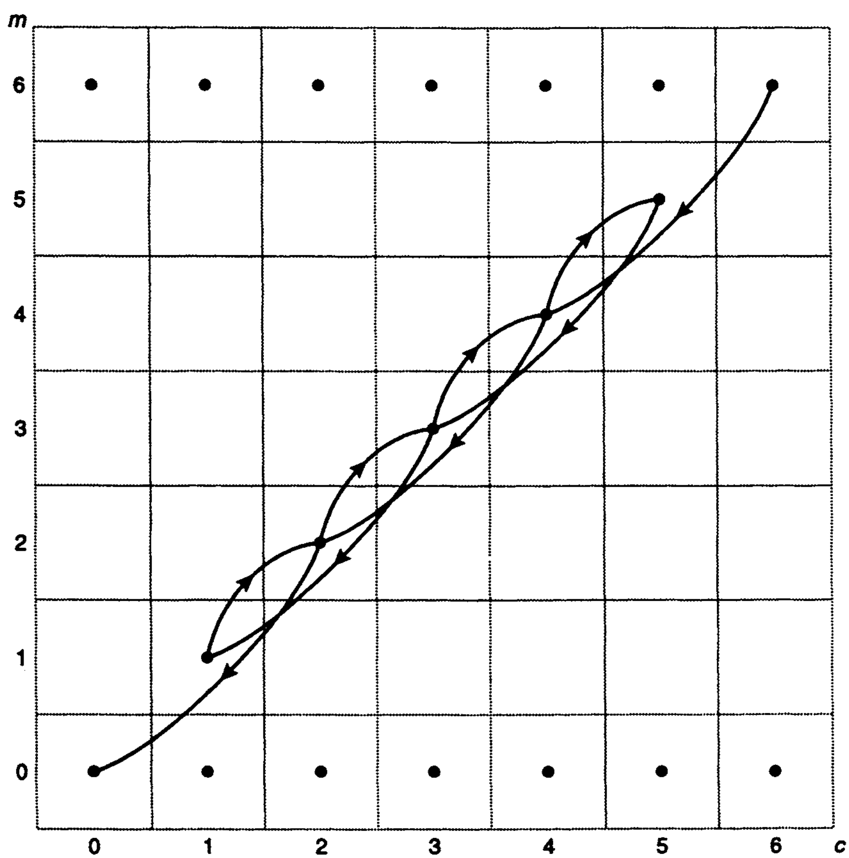


Figure 44

pose now that the boat is enlarged to hold three passengers and that on the boat, as on the bank, the cannibals must not outnumber the missionaries. Under these conditions all eight can cross safely in as few as nine steps. Five cannibals and five missionaries can also cross in a boat that holds three passengers (in eleven steps), but six cannibals and six missionaries cannot.

It is easy to see that given a boat holding four or more passengers any group evenly divided between cannibals and missionaries can be safely transported across the river. One cannibal and one missionary simply do all the rowing, transporting the others one cannibal-missionary pair at a time until the job is done. Now let n be the number of cannibals (or missionaries). If the boat holds just four passengers, the problem is solvable in $2n - 3$ steps. If the boat holds an even number of passengers that is greater than 4, more than one cannibal-missionary pair can of course be taken each time. The technique of always keeping the same number of cannibals and missionaries on both sides of the river is diagrammed as a braided pattern along the diagonal of the matrix of the problem as is shown in Figure 45. This nine-step digraph solves the cannibal-missionary problem when n equals 6 and the boat holds four passengers.

Figure 45



When the capacity of the boat is an even number greater than or equal to 4, the diagonal method always gives the best solution. If the number of cannibals n is just one more than the capacity of the boat, which is an even number greater than 4, then there is always a five-step minimum solution. Actually the diagonal method is more powerful than this last case implies. With a boat that holds an even number greater than 4 it will always provide a five-step minimum solution for any case from $b + 1$ cannibals through $(3b/2) - 2$ cannibals, where b is the capacity of the boat.

If the number of passengers the boat can hold is odd, moving down the diagonal does not always give the best answer. For example, if n equals 6 and the boat holds five, the diagonal method gives the same nine-move solution shown in Figure 45, but the problem also has a seven-step solution. More generally, if the boat holds an odd number of passengers that is greater than three and one less than n, there always is a minimum solution in seven moves. Can you find one of many seven-step solutions for six cannibals and six missionaries crossing the river in a boat that holds five passengers? This is the simplest of an infinity of examples in which, for a boat with an odd capacity, there is a procedure superior to the diagonal procedure. (I am ignoring here the trivial cases of a boat with an odd capacity of one or three, where the diagonal method will not work at all.) The next simplest case is the one where n equals 10 and the boat holds seven passengers.

The digraph method can be applied to almost any kind of river-crossing problem. One famous problem, which goes back at least to the eighth century, concerns three jealous husbands and their wives, who want to cross a river in a boat that holds two passengers. How can this goal be accomplished so that a wife is never alone with a man who is not her husband? If you construct the digraph for the problem, you may be surprised to discover that it is solved by the same four walks as the classic cannibal–missionary problem and has no other solutions.

The only difference—and this applies also to generalizations of the jealous-husband variant of the puzzle—is that the pairings of individual men and women have to be manipulated to meet conditions not essential to the cannibal-missionary version.

Many puzzle books include more exotic variations of the cannibal-missionary problem. For example, in some cases only certain people may be able to row. (In the classic problem if only one cannibal and one missionary can row, the solution requires 13 crossings.) The boat may also have a minimum capacity (greater than one) as well as a maximum capacity. Or missionaries may outnumber cannibals and be safe only if they outnumber them at all times. An island in the river may also be employed as a stopover spot, and certain pairs of individuals may be singled out as being too incompatible to be left alone together.

An ancient problem of this last type (it too can be traced back to the eighth century) is about a man who wants to ferry a wolf, a goat, and a cabbage across a river in a boat that allows him to take only one of them at a time. He cannot leave the wolf alone with the goat or the goat alone with the cabbage. In this case there are two minimal solutions, each of which requires seven trips. One of these solutions is shown in Figure 46, taken from *Moscow Puzzles*, by Boris A. Kordemsky (Charles Scribner's Sons, 1972). Interested readers will find a good selection of such river-crossing problems in books by the British puzzle expert Henry Ernest Dudeney.

I have space for one more digraph puzzle. Paul Erdős has shown that on a complete digraph for n points, when n is less than 7, it is not possible to place arrowheads so that for any two specified points it is always possible to get to each point in one step from some third point. Figure 47 shows a complete graph for seven points. Think of the points as towns joined by one-way roads. Your task is to label each road with an arrowhead so that for any specified pair of towns there is a third

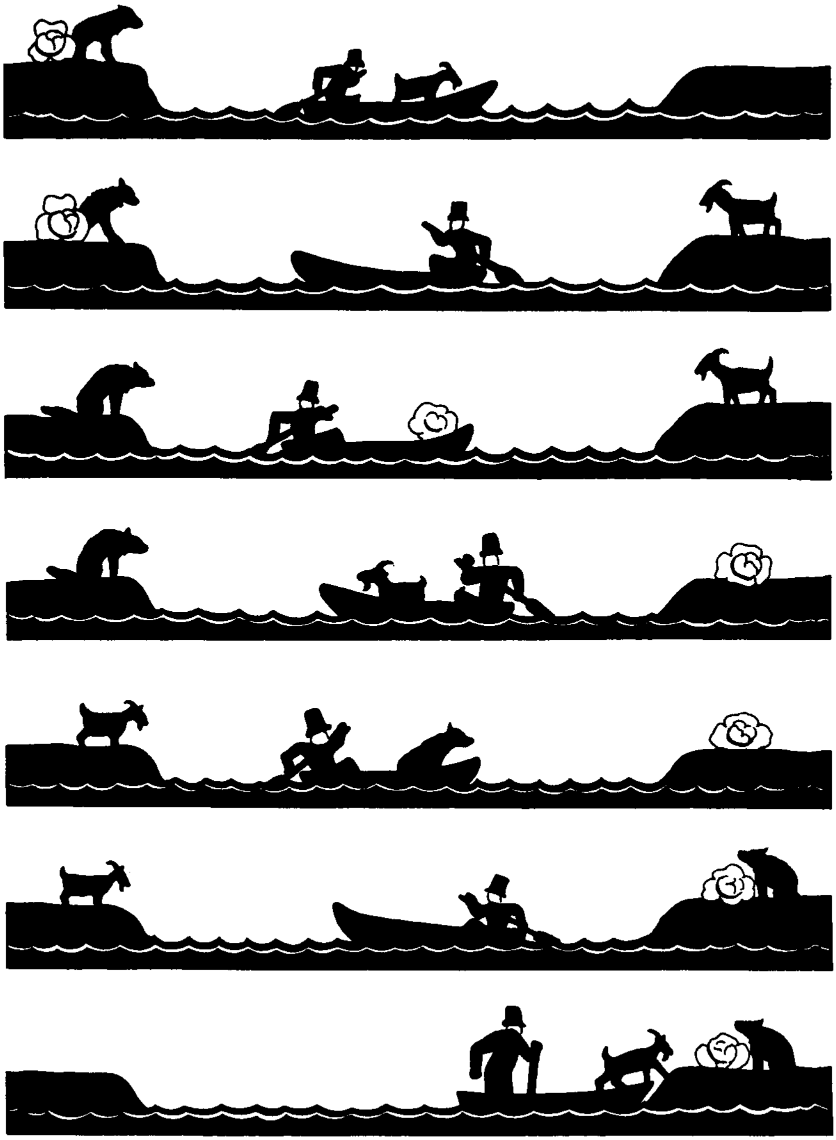


Figure 46

town from which you can drive directly to each of the other two. There is only one solution.

Graphs of this sort are usually called *tournament graphs* because the points can represent players, and the arrows show who beats who. In

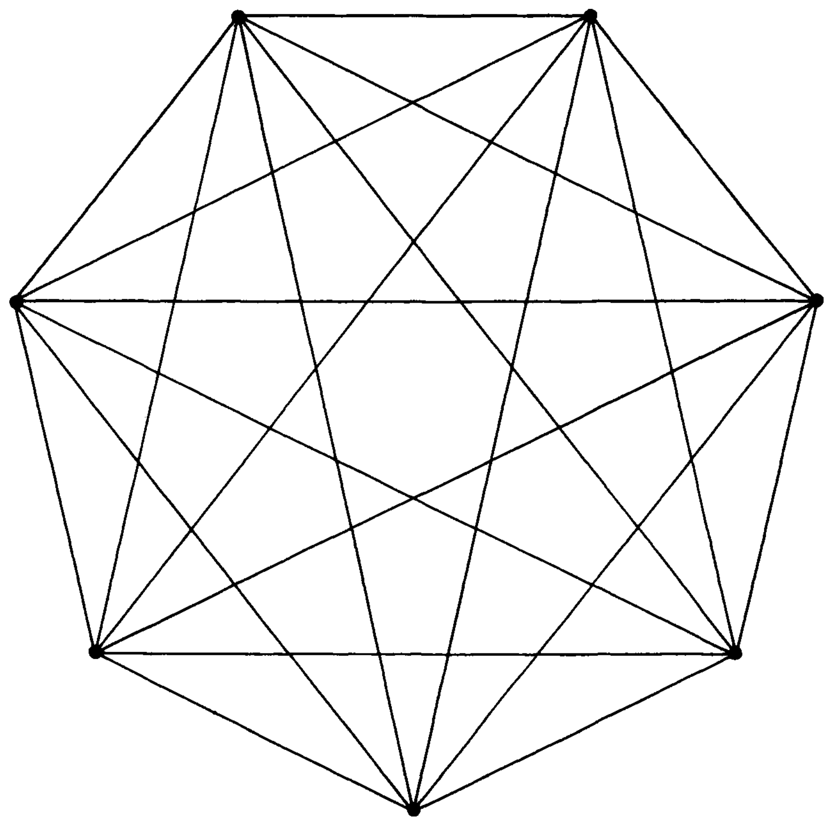


Figure 47

this interpretation, no graph with fewer than seven points can show that for any two players there is always a third person who beats them both. The seven-point graph is the smallest in which this can be the case. It is nontransitive. There is no "best" player because each player can be defeated by another person.

Answers

The unique Hamiltonian circuit is found by starting at A and following a directed path that spells AMBIDEXTROUS. One more step joins S to A, honoring *Scientific American.*

Figure 48 shows a digraph for one of many seven-step solutions to

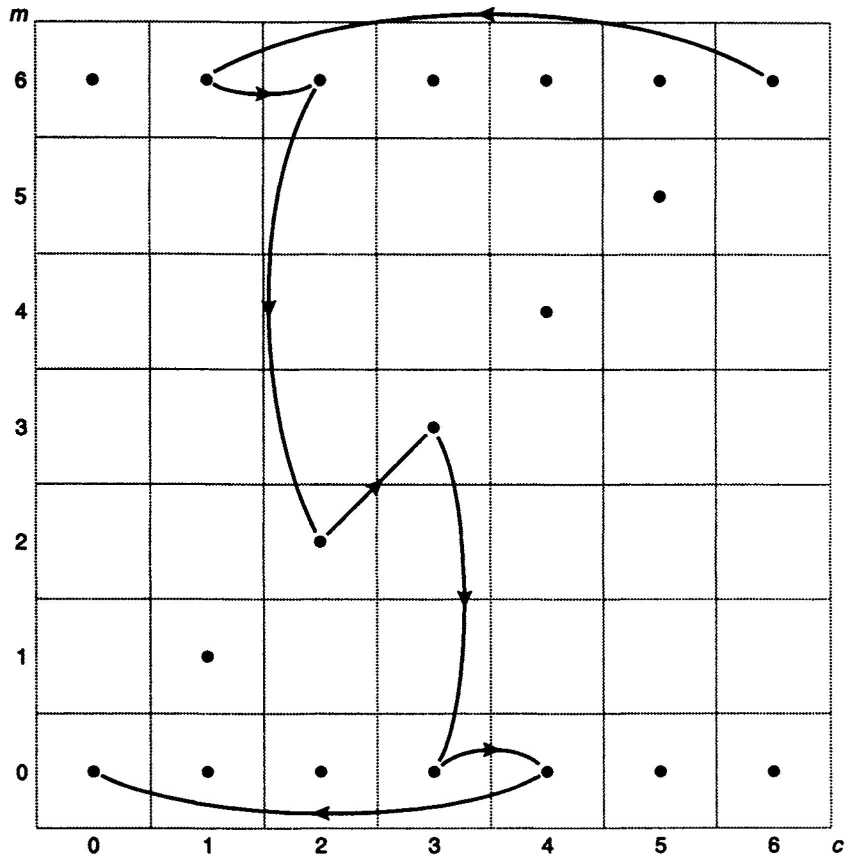


Figure 48

the problem of six missionaries and six cannibals who want to cross a river safely in a boat that holds five.

The Paul Erdös problem is solved by placing arrows on the complete graph for seven points as is shown in Figure 49. Of course, the points and their connecting lines can be permuted in any way to provide solutions that do not appear in this symmetrical form, but all such solutions are topologically the same. See "On a General Problem in Graph Theory," by Paul Erdös in *The Mathematical Gazette* (Vol. 47, No. 361, pages 220–223; October 1963).

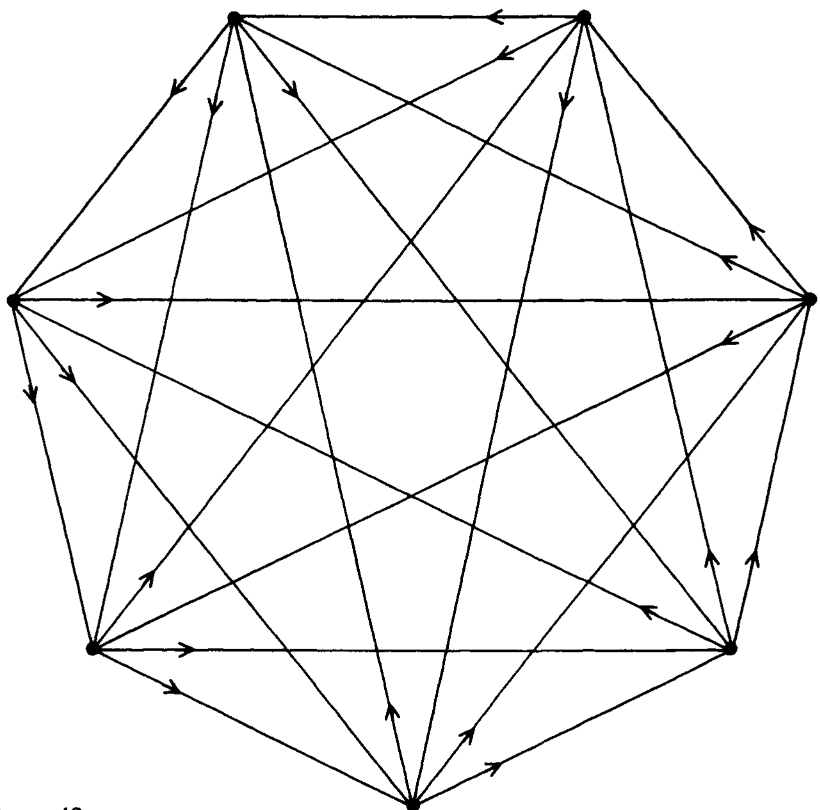


Figure 49

ADDENDUM

Frank Harary was the first to define the distance matrix, the reachability matrix, and the detour matrix, as well as the first to introduce many other graph theory terms that are now standard such as strongly and weakly connected digraphs. This is why Gerhard Ringel, reviewing Harary's classic textbook *Graph Theory*, called him the graph theory Pope. It is because Harary gives the word!

For many years Harary has been inventing and solving two-person games played on graphs. He calls a game in which a defined goal is reached by the winner an "achievement game." If the first person forced to reach the goal is the loser, it is an "avoidance game." His massive work on both types of games remains, alas, unpublished except for occasional papers.

An example of one of Harary's digraph games, which he described to me in a 1980 letter, is a game he calls Kingmaker. Every tournament graph—a complete digraph, every pair of points joined by an arc or directed line—has at least one point called the King that has a distance of 1 or 2 from every other point. This is sometimes known as the King Chicken Theorem.

Kingmaker starts with an undirected complete graph of n points. The first player draws an arrow on any line. Of course it doesn't matter what line he selects because all are alike for symmetry reasons. (Harary suggests that the second player and all onlookers shout "Shrewd move!" after this first arrow is drawn.) The winner is the first to produce a King, in this case a point with a distance of 1 or 2 from all points joined directly from the King by arrows. This usually occurs before all the lines are oriented. In the avoidance game, the player forced to make a King loses. This tends to occur after almost all lines have an arrow.

Steve Maurer, at Swarthmore College, has done much of the work on theorems involving Kings. Every tournament—that is, every complete digraph—must have at least one King, but no such graph can have exactly two Kings. If there are two, there must be a third. Interpreting the points as chickens, a chicken who pecks every other chicken must be the group's only King. A chicken pecked by all the others cannot be a King. A graph with an odd number of points (chickens) can consist entirely of Kings. These theorems provided an amusing page of brain teasers titled "Chicken a la King," by Maxwell Carver (a pseudonym of Joel Spencer), in *Discover*, March 1988, page 96.

Digraphs furnish a neat, little known method for diagramming problems in the propositional calculus of formal logic. See "The Propositional Calculus with Directed Graphs," on which Harary and I collaborated (giving me my first Erdös number of 2). It appeared in Cambridge University's undergraduate mathematics journal *Eureka*, March 1988, pages 34–40. The technique is also covered in an appendix added to the second edition (University of Chicago Press, 1982) of my *Logic Machines and Diagrams.*

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