Acyclic Edge Coloring of Graphs

A THESIS
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by

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To

My Family

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Abstract

A proper *edge coloring* of $G = (V, E)$ is a map $c : E \rightarrow C$ (where C is the set of available $colors$) with $c(e) \neq c(f)$ for any adjacent edges e,f. The minimum number of colors needed to properly color the edges of G, is called the chromatic index of G and is denoted by $\chi'(G)$.

A proper edge coloring c is called acyclic if there are no bichromatic cycles in the graph. In other words an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in G. The *acyclic edge chromatic number* (also called *acyclic chromatic index*), denoted by $a'(G)$, is the minimum number of colors required to acyclically edge color G.

The primary motivation for this thesis is the following conjecture by Alon, Sudakov and Zaks [7] (and independently by Fiamcik [22]):
Acyclic Edge Coloring Conjecture: For any graph G, $a'(G) \leq \Delta(G) + 2$.

The following are the main results of the thesis:

1. From a result of Burnstein [16], it follows that any subcubic graph can be acyclically edge colored using at most 5 colors. Skulrattankulchai [38] gave a polynomial time algorithm to color a subcubic graph using $\Delta + 2 = 5$ colors. We proved that any non-regular subcubic graph can be acyclically colored using only 4 colors. This result is tight. This also implies that the fifth color, when needed is required only for one edge.

2. Let G be a connected graph on n vertices, $m \leq 2n - 1$ edges and maximum degree $\Delta \leq 4$, then $a'(G) \leq 6$. This implies that graph of maximum degree 4 are acyclically edge colorable using at most 7 colors.

3. The earliest result on acyclic edge coloring of 2-degenerate graphs was by Caro and Roditty [17], where they proved that $a'(G) \leq \Delta + k - 1$, where k is the maximum edge connectivity, defined as $k = \max_{u,v \in V(G)} \lambda(u, v)$, where $\lambda(u, v)$ is the edge- connectivity of the pair u,v. Note that here k can be as high as Δ. Muthu,Narayanan and

Subramanian [34] proved that $a'(G) \leq \Delta+1$ for outerplanar graphs which are a subclass of 2-degenerate graphs and posed the problem of proving the conjecture for 2-degenerate graphs as an open problem. In fact they have also derived an upper bound of $\Delta + 1$ for series-parallel graphs [35], which is a slightly bigger subclass of 2-degenerate graphs. We proved that 2-degenerate graphs are $\Delta + 1$ colorable.

4. Fiedorowicz, Hauszczak and Narayanan [24] gave an upper bound of $2\Delta + 29$ for planar graphs. Independently Hou, Wu, GuiZhen Liu and Bin Liu [29] gave an upper bound of $max(2\Delta - 2, \Delta + 22)$. We improve this upper bound to $\Delta + 12$, which is the best known bound at present.

5. Fiedorowicz, Hauszczak and Narayanan [24] gave an upper bound of $\Delta + 6$ for triangle free planar graphs. We improve the bound to $\Delta + 3$, which is the best known bound at present.

6. We have also worked on lower bounds. Alon et. al. [7], along with the acyclic edge coloring conjecture, also made an auxiliary conjecture stating that Complete graphs of $2n$ vertices are the only class of regular graphs which require $\Delta+2$ colors. We disproved this conjecture by showing infinite classes of regular graphs other than Complete Graphs which require $\Delta + 2$ colors.

Apart from the above mentioned results, this thesis also contributes to the acyclic edge coloring literature by introducing new techniques like Recoloring, Color Exchange (exchanging colors of adjacent edges), circular shifting of colors on adjacent edges (derangement of colors). These techniques turn out to be very useful in proving upper bounds on the acyclic edge chromatic number.

Contents

Chapter 1

Introduction

The origins of graph theory are humble, even frivolous. Whereas many branches of mathematics were motivated by the fundamental problems of calculation, motion, and measurement, the problems which led to the development of graph theory were often little more than puzzles, designed to testify the ingenuity rather than to stimulate the imagination. But despite the apparent triviality of such problems, they captured the interest of mathematicians, with the result that graph theory has become a subject rich in theoretical results of a surprising variety and depth [13].

Little did Francis Gutherie knew that his simple observation would catapult the evolution of graph theory to what it is now and much little did he knew that his observation (conjecture) would take mathematicians more than 100 years to prove! Yes, we are talking about the most famous problem of graph theory- The Four Color Problem (now Theorem): Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?

In terms of graphs[1] Four Color Problem is equivalent to asking whether a planar graph (i.e., a graph drawn on a plane such that none of its edges cross) can be four colored such that adjacent vertices get different colors.

Tait came up with an equivalent statement to the Four Color Problem in terms of edge coloring (coloring of the edges of a graph such that adjacent edges get different colors) of a cubic graph- Is every bridgeless (a bridge is a cut edge) cubic planar graph edge colorable using three colors? Though he gave a proof, it was a wrong one. But this introduced edge coloring

[1]The term $graph$ as we use now was coined for the first time by $Sylvester$ in a paper published in 1878 in $Nature$

as a problem. A coloring of vertices (edges) is called proper if adjacent vertices (edges) get different colors. Now if we examine carefully even-cycles could be colored using 2 colors, where as odd-cycles require 3 colors. And hence the presence of odd-cycles in a graph increase the number of colors required to color the graph properly. Researchers thought what would happen if we restrict all cycles (odd or even) to use at least three colors in addition of being properly colored. This is how the concept of acyclic coloring originated. Acyclic coloring is a proper coloring of the vertices (edges) such that there exists no bichromatic (2-colored) cycle in the graph. The concept of *acyclic coloring* of a graph was introduced by Grünbaum [27]. The *acyclic chromatic index* and its vertex analogue can be used to bound other parameters like *oriented chromatic number* and *star chromatic number* of a graph, both of which have many practical applications, for example, in wavelength routing in optical networks ([9], [31]). Now let us look at the acyclic edge coloring of graphs throughout this thesis.

1.1 Basics and Notations

A graph is a pair $G = (V, E)$ of sets satisfying $E \subseteq V \times V$; thus, the elements of E are 2-element subsets of V. The elements of V are the $vertices$ (or $nodes$, or $points$) of the graph G, the elements of E are its $edges$ (or $lines$). The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. How these dots and lines are drawn is considered irrelevant: all that matters is the information, which pairs of vertices form an edge and which do not.

The vertex set of a graph G is referred to as $V(G)$ and its edge set as $E(G)$. Two vertices $x, y \in G$ are $adjacent$ or $neighbours$ if they have an edge between them, i.e. $(x, y) \in E(G)$. Then the edge (x, y) is said to be $incident$ on vertices x and y. If all the vertices of a graph are pairwise adjacent, then the graph is known as a $complete\ graph$. A complete graph on n vertices is denoted by K_n. The $degree$ of a vertex v in graph G is the number of edges incident on v and is denoted by $deg_G(v)$. The number $\delta(G) = min\{deg_G(v)|v \in V(G)\}$ is the $minimum\ degree$ of G and the number $\Delta(G) = max\{deg_G(v)|v \in V(G)\}$ is its $maximum\ degree$. $N_G(u)$ denotes all the neighbours of vertex u in G. Whenever the underlying graph G is clear from the context, we omit the subscript and use $deg(u)$ and $N(u)$ to denote the degree and neighbours of u respectively.

A graph H is a $subgraph$ of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is said to be an $induced\ subgraph$ of graph G if it is a subgraph of G and for every pair of vertices

$x, y \in V(H)$, edge $(x, y) \in E(H)$ if and only if $(x, y) \in E(G)$. A $matching$ is a set of independent edges in the graph. A matching is called $perfect$ if all the vertices are present in the matching. A graph is called $finite$ if the number of vertices and edges in the graph are finite. A graph is called $simple$ if between each pair of vertices, there is at most one edge and no loops. A graph is $directed$ if the edges have direction. Here the edge (x, y) is different from edge (y, x). If the edges are not directed, then the graph is $undirected$. All graphs considered in this thesis are finite, simple and undirected. See [18] for further notations and definitions.

Definition 1.1. *A proper* edge coloring *of* $G = (V, E)$ *is a map* $c : E \to C$ *(where* C *is the set of available* $colors$ *) with* $c(e) \neq c(f)$ *for any adjacent edges* e,f. *The minimum number of colors needed to properly color the edges of* G*, is called the chromatic index of* G *and is denoted by* $\chi'(G)$.

Definition 1.2. *A proper edge coloring c is called acyclic if there are no bichromatic cycles in the graph. In other words an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in* G. *The* acyclic edge chromatic number *(also called* acyclic chromatic index*), denoted by* $a'(G)$*, is the minimum number of colors required to acyclically edge color* G.

1.2 Motivation

The primary motivation for this thesis is the following conjecture:

Acyclic Edge Coloring Conjecture: For any graph G, $a'(G) \leq \Delta + 2$.

The conjecture is due to Fiamcik [20] (Alon, Sudakov and Zaks [7] also independently conjectured the same). The conjecture is open and is proved true for only few special classes of graphs.

The problem of finding $a'(G)$ is intimately related to other well known conjectures in graph theory. We briefly comment about them here.

1. **Perfect 1-factorization Conjecture:** For any $n \geq 2$, K_{2n} can be decomposed into $2n-1$ perfect matchings such that the union of any two matchings forms a hamiltonian cycle of K_{2n}.

Apart from proving that the conjecture holds for certain values of n, for instance, if n is prime (see [13] for a summary of the known cases), this conjecture of Kotzig and others is still open. Alon et. al. in [7] observed the equivalence of this conjecture to determining $a'(G)$ of the complete graph.

If such a decomposition of K_{2n+2} (called a perfect 1-factorization) exists, then by coloring every perfect matching using a different color and removing one vertex we obtain an acyclic edge coloring of K_{2n+1} with $2n + 1 = \Delta(K_{2n+1} + 1$ colors. Such a coloring is best possible for K_{2n+1} since it is $2n$-regular (It is easy to prove that any d-regular graph requires at least $d + 1$ colors. A proof is given in one of the later chapters in the thesis).

A decomposition of K_{2n+1} into $2n + 1$ matchings each having n edges, such that the union of any two matchings forms a Hamiltonian path of K_{2n+1} is called a *perfect near-1-factorization*. As shown above, if K_{2n+2} has a perfect 1-factorization then K_{2n+1} has a perfect near-1-factorization, which is turn implies that $a'(K_{2n+1}) = 2n + 1$. It is easy to see that the converse is also true: if K_{2n+1} has an acyclic edge coloring with $2n + 1$ colors, then this coloring corresponds to a perfect near-1-factorization of K_{2n+1} which implies that K_{2n+2} has a perfect 1-factorization. Therefore the following statements are equivalent:

- K_{2n+2} has a perfect 1-factorization.
- K_{2n+1} has a perfect near-1-factorization.
- $a'(K_{2n+1}) = 2n + 1$.

2. The linear arboricity of a graph, denoted by $la(G)$, is the minimum number of linear forest into which the edges of the graph can be partitioned. It was shown by Akiyama, Exoo and Harary [3] that $la(G) = 2$ when G is cubic, and they conjectured that for every d-regular graph,

   Linear Arboricity Conjecture: $la(G) = \lceil (d + 1)/2 \rceil$.

   This conjecture can be restated in terms of Maximum degree Δ as follows: For any graph $\lceil (\Delta + 1)/2 \rceil$. Akiyama, Exoo and Harary ([3], [4]) proved the conjecture for complete graphs, complete bipartite graphs, trees and graphs with $\Delta = 3, 4$. Alon [5] proved that, for every $\epsilon > 0$, $la(G) \leq (\frac{1}{2} + \epsilon)\Delta$ for every graph G with sufficiently large

Δ; moreover, the conjecture for every graph G with an even (or odd) maximum degree Δ and with girth $g \geq 50\Delta$ ($g \geq 100\Delta$). See [2], [19], [28] for more details.

As we know any two color classes of an acyclic edge coloring induce a linear forest in the graph and thus it is obvious that $la(G) \leq \lceil a'(G)/2 \rceil$. If we can show that for a given class of graphs, $a'(G) \leq \Delta + 1$, then it would immediately follow that the conjecture for linear arboricity is true for that class of graphs.

1.3 History and Literature Survey

The concept of *acyclic coloring* of a graph was introduced by Grünbaum [27]. He introduced acyclic vertex coloring and conjectured that the vertices of every planar graph can be colored acyclically using 5 colors. Later Borodin [14] proved it.

Acyclic Edge Coloring was studied by Fiamcik [20] and he proposed the acyclic edge coloring conjecture in 1978. He solved the conjecture for subcubic graphs. His papers were not available in English till recently and hence was unknown. Alon, McDiarmid and Reed [6] introduced it independently and using probabilistic methods proved that $a'(G) \leq 64\Delta$. They also mentioned that the constant 64 could be improved with more careful application of the Lovasz Local Lemma. Later Molloy and Reed showed that $a'(G) \leq 16\Delta$. This is the best known bound currently for arbitrary graphs. Muthu, Narayanan and Subramanian [32] proved that $a'(G) \leq 4.52\Delta$ for graphs G of girth at least 220 (*Girth* is the length of a shortest cycle in a graph). All the above results use probabilistic methods. The best known constructive bound is by Subramanian [39] who showed that $a'(G) \leq 5\Delta(\log \Delta + 2)$.

Though the best known upper bound for general case is far from the conjectured $\Delta + 2$, the conjecture has been shown to be true for some special classes of graphs. Alon, Sudakov and Zaks [7] proved that there exists a constant k such that $a'(G) \leq \Delta + 2$ for any graph G whose girth is at least $k\Delta \log \Delta$. They also proved that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. This result was improved by Nešetřil and Wormald [36] who showed that for a random Δ-regular graph $a'(G) \leq \Delta + 1$. Muthu, Narayanan and Subramanian proved the conjecture for grid-like graphs [33] and outer planar graphs [34]. In fact they gave a better bound of $\Delta + 1$ for those classes of graphs. From Burnstein's [16] result it follows that the conjecture is true for subcubic graphs. Skulrattankulchai [38] gave a polynomial time algorithm to color a subcubic graph using $\Delta + 2 = 5$ colors.

Determining $a'(G)$ is a hard problem both from a theoretical and from an algorithmic point of view. Even for the simple and highly structured class of complete graphs, the value of $a'(G)$ is still not determined exactly. The difficulty in determining $a'(G)$ for complete graphs could be observed by its equivalence to the *Perfect 1-factorization Conjecture*. It has also been shown by Alon and Zaks [8] that determining whether $a'(G) \leq 3$ is NP-complete for an arbitrary graph G.

A generalization of the acyclic edge chromatic number has also been studied. The *r-acyclic edge chromatic number* $a'_r(G)$ is the minimum number of colors sufficient to color the edges of the graph G such that every cycle C of G has at least min$\{|C|,r\}$ colors (see [25], [26]).

1.4 Thesis Overview

In Chapter 2, we develop the preliminaries which are extensively used in the proofs of our theorems.

Chapters 3-7 deal with the upper bound for special classes of graphs. Chapter 3 gives a tight bound for subcubic graphs, while Chapter 4 gives an upper bound for graphs with maximum degree 4. Chapter 5 shows that 2-degenerate graphs are $\Delta + 1$ acyclically edge colorable and this bound is tight. In Chapter 6, we look at the acyclic edge coloring of Planar graphs. We obtain an upper bound of $\Delta + 12$ for planar graphs. In Chapter 7, we look at triangle free planar graphs where we reduce the upper bound to $\Delta + 3$.

Chapter 8 deals with the lower bounds. Alon et.al. conjectured that $G = K_{2n}$ might be the only regular graphs which have $a'(G) \geq \Delta + 2$. We disprove this conjecture. Here we consider dense graphs which give us a lower bound of $\Delta + 2$. Also we give the exact bound for $K_{p,p}$, where p is an odd prime.

Chapter 2

Preliminaries

In this chapter, we shall look at the definitions, facts and lemmas that are used in the later chapters. The reader is advised to go through it carefully since these things are extensively used in the proofs later.

Let $G = (V, E)$ be a simple, finite and connected graph of n vertices and m edges. Let $x \in V$. Then $N_G(x)$ will denote the neighbors of x in G. For an edge $e \in E$, $G - e$ will denote the graph obtained by deletion of the edge e. For $x, y \in V$, when $e = (x, y) = xy$, we may use $G - \{xy\}$ instead of $G - e$. Let $c : E \rightarrow \{1, 2, \ldots, k\}$ be an *acyclic edge coloring* of G. For an edge $e \in E$, $c(e)$ will denote the color given to e with respect to the coloring c. For $x, y \in V$, when $e = (x, y) = xy$ we may use $c(x, y)$ instead of $c(e)$. For $S \subseteq V$, we denote the induced subgraph on S by $G[S]$.

Partial Coloring: Let H be a subgraph of G. Then an acyclic edge coloring c' of H is also a partial coloring of G. Note that H can be G itself. Thus a coloring c of G itself can be considered a partial coloring. A coloring c of G is said to be a proper partial coloring if c is proper. A proper partial coloring c is called acyclic if there are no bichromatic cycles in the graph. Sometimes we also use the word valid coloring instead of acyclic coloring. Note that with respect to a partial coloring c, $c(e)$ may not be defined for an edge e. So, whenever we use $c(e)$, we are considering an edge e for which $c(e)$ is defined, though we may not always explicitly mention it.

Let c be a partial coloring of G. We denote the set of colors in the partial coloring c by $C = \{1, 2, \ldots, \Delta + 1\}$. For any vertex $u \in V(G)$, we define $F_u(c) = \{c(u, z) | z \in N_G(u)\}$. For an edge $ab \in E$, we define $S_{ab}(c) = F_b(c) - \{c(a, b)\}$. Note that $S_{ab}(c)$ need not be the same as $S_{ba}(c)$. We will abbreviate the notation to F_u and S_{ab} when the coloring c is understood from the context.

Let $G = (V, E)$ be a graph on m edges where $m \geq 1$. We will remove an edge e from G and get a graph $G' = (V, E')$. By the minimality of G, the graph G' will have an acyclic edge coloring $c : E' \rightarrow \{1, 2, \ldots, \Delta + 1\}$. Our intention will be to extend the coloring c of G' to G by assigning an appropriate color for the edge e thereby contradicting the assumption that G is a minimum counter example.

The following definitions arise out of our attempt to understand what may prevent us from extending a partial acyclic coloring of $G - e$ to G.

Maximal bichromatic Path: Consider the subgraph induced by any two colors α and β with respect to any proper coloring c. The (α, β)-subgraph consists of even cycles, bichromatic paths of length at least two, isolated edges and isolated vertices. Now when we say maximal bichromatic path, we only concentrate on bichromatic paths of length at least two, ignoring the even bichromatic cycles, isolated edges and isolated vertices. Thus an (α,β) maximal bichromatic path with respect to a proper coloring c of G is a path-component of the (α, β)-subgraph that consists of at least two edges. An (α,β,a,b) maximal bichromatic path is an (α,β) maximal bichromatic path which starts at the vertex a with an edge colored α and ends at b. We emphasize that the edge of the (α,β,a,b) maximal bichromatic path incident on vertex a is colored α and the edge incident on vertex b can be colored either α or β. Thus the notations (α,β,a,b) and (α,β,b,a) have different meanings. Also note that any maximal bichromatic path will have at least two edges. The following fact is obvious from the definition of acyclic edge coloring:

Fact 2.1. *Given a pair of colors α and β of a proper coloring c of G, there can be at most one maximal (α,β) bichromatic path containing a particular vertex v, with respect to c.*

Definition 2.2. *A color $\alpha \neq c(e)$ is a* candidate *for an edge* e *in G with respect to a partial coloring c of G if none of the adjacent edges of* e *are colored α. A candidate color α is* valid *for an edge* e *if assigning the color α to* e *does not result in any bichromatic cycle in G.*

Let $e = (a, b)$ be an edge in G. Note that any color $\beta \notin F_a \cup F_b$ is a candidate color for the edge ab in G with respect to the partial coloring c of G. A sufficient condition for a candidate color being valid is captured in the Lemma below:

Lemma 2.3. *A candidate color for an edge $e = uv$, is valid if $(F_u \cap F_v) - \{c(u,v)\} = (S_{uv} \cap S_{vu}) = \emptyset$.*

Proof: Any cycle containing the edge uv will also contain an edge incident on u (other than uv) as well as an edge incident on v (other than uv). Clearly these two edges are colored differently since $(S_{uv} \cap S_{vu}) = \emptyset$. Thus the cycle will have at least 3 colors and therefore any of the candidate colors for the edge uv is valid. ■

Now even if $S_{ab} \cap S_{ba} \neq \emptyset$, a candidate color β may be valid. But if β is not valid, then what may be the reason? It is clear that color β is not $valid$ if and only if there exists $\alpha \neq \beta$ such that a (α,β) bichromatic cycle gets formed if we assign color β to the edge e. In other words, if and only if, with respect to coloring c of G there existed a (α,β,a,b) maximal bichromatic path with α being the color given to the first and last edge of this path. Such paths play an important role in our proof. We call them $critical\ paths$. It is formally defined below:

Definition 2.4. ***Critical Path:*** *Let $ab \in E$ and c be a partial coloring of G. Then a $(\alpha, \beta$,a,b$)$ maximal bichromatic path which starts out from the vertex a via an edge colored α and ends at the vertex b via an edge colored α is called an (α, β, ab) critical path. Note that any critical path will be of odd length. Moreover the smallest length possible is three.*

Remark: When we define critical paths, we always keep the graph G in mind even though we are dealing with coloring of a subgraph. Thus when we say ab is an edge, its an edge in G.

An obvious strategy to extend a valid partial coloring c of G would be to try to assign one of the candidate colors to an uncolored edge e. The condition that a candidate color being not valid for the edge e is captured in the following fact:

Fact 2.5. *Let c be a partial coloring of G. A candidate color β is not $valid$ for the edge $e = (a, b)$ if and only if $\exists \alpha \in S_{ab} \cap S_{ba}$ such that there is a (α, β, ab) critical path in G with respect to the coloring c.*

Definition 2.6. ***Actively Present:*** *Let c be a partial coloring of G. Let $a \in N_G(x)$ and let $c(x, a) = \alpha$. Let $\beta \in S_{xa}$. Color β is said to be* actively present *in a set S_{xa} with respect to the edge xy, if there exists a (α, β, xy) critical path. When the edge xy is understood in the* *context, we just say that β is actively present in S_{xa}.*

If all the candidate colors turn out to be $invalid$, we try to $slightly\ modify$ the partial coloring c in such a way that with respect to the modified coloring, one of the candidate colors becomes valid. An obvious way to modify is to recolor an edge so that some critical paths are $broken$ and a candidate color becomes valid. Sometimes we resort to a slightly more sophisticated strategy to modify the coloring namely $color\ exchange$ defined below:

Color Exchange: Let c be a partial coloring of G. Let $u, i, j \in V(G)$ and $ui, uj \in E(G)$. We define $Color\ Exchange$ with respect to the edge ui and uj, as the modification of the current partial coloring c by exchanging the colors of the edges ui and uj to get a partial coloring c', i.e., $c'(u,i) = c(u,j)$, $c'(u,j) = c(u,i)$ and $c'(e) = c(e)$ for all other edges e in G. The color exchange with respect to the edges ui and uj is said to be proper if the coloring obtained after the exchange is proper. The color exchange with respect to the edges ui and uj is $valid$ if and only if the coloring obtained after the exchange is acyclic. The following fact is obvious:

Fact 2.7. *Let c' be the partial coloring obtained from a valid partial coloring c by the color exchange with respect to the edges ui and uj. Then the partial coloring c' will be proper if and only if $c(u,i) \notin S_{uj}$ and $c(u,j) \notin S_{ui}$.*

The color exchange is useful in breaking some critical paths as is clear from the following lemma:

Lemma 2.8. *Let c be a partial coloring of G and let $u, i, j, a, b \in V(G)$, $ui, uj, ab \in E$. Also let $\{\lambda, \xi\} \in C$ such that $\{\lambda, \xi\} \cap \{c(u,i), c(u,j)\} \neq \emptyset$ and $\{i, j\} \cap \{a, b\} = \emptyset$. Suppose there exists an (λ,ξ,ab) critical path that contains vertex u, with respect to a valid partial coloring c of G. Let c' be the partial coloring obtained from c by the color exchange with respect to the edges ui and uj. If c' is proper, then there will not be any (λ,ξ,ab) critical path in G with respect to the partial coloring c'.*

Proof: Firstly, $\{\lambda, \xi\} \neq \{c(u,i), c(u,j)\}$. This is because, if there is a (λ,ξ,ab) critical path that contains vertex u, with respect to a valid partial coloring c of G, then it has to contain the edge ui and uj. Since $i \notin \{a, b\}$, vertex i is an internal vertex of the critical path which implies that both the colors λ and ξ (that is $c(u,i)$ and $c(u,j)$) are present at vertex i. That means $c(u,j) \in S_{ui}$ and this contradicts $Fact$ 2.7, since we are assuming that the color exchange is proper. Thus $\{\lambda, \xi\} \neq \{c(u,i), c(u,j)\}$.

Now let P be the (λ,ξ,ab) critical path with respect to the coloring c. Without loss of generality assume that $\gamma = c(u,i) \in \{\lambda, \xi\}$. Since vertex u is contained in path P, by the maximality of the path P, it should contain the edge ui since $c(u,i) = \gamma \in \{\lambda, \xi\}$. Let us assume without loss of generality that path P starts at vertex a and reaches vertex i before it reaches vertex u. Now after the color exchange with respect to the edges ui and uj, i.e., with respect to the coloring c', there will not be any edge adjacent to vertex i that is colored γ. So if any (λ,ξ) maximal bichromatic path starts at vertex a, then it has to end at vertex i. Since $i \neq b$, by $Fact$ 2.1 we infer that the (λ,ξ,ab) critical path does not exist. ■

Chapter 3

Subcubic Graphs

A graph is called subcubic if the maximum degree of the graph is three. In this chapter we will look at the acyclic edge coloring of subcubic graphs.

3.1 Previous Results on Subcubic Graphs

Burnstein's [16] proved that if $\Delta(G) \leq 4$, G can be acyclically vertex colored using at most 5 colors. The line graph of any graph of maximum degree at most 3 (i.e., a subcubic graph) has maximum degree at most 4. Since acyclic edge coloring of a graph is nothing but the acyclic vertex coloring of its line graph, it follows that any subcubic graph can be acyclically edge colored using at most 5 colors. Skulrattankulchai [38] gave a polynomial time algorithm to color a subcubic graph using $\Delta + 2 = 5$ colors. Alon,Sudakov and Zaks mentioned in [7] that they have also found a polynomial time algorithm for the same.

3.2 The Theorem

Theorem 3.1. *Let G be a non-regular connected graph of maximum degree 3, then $a'(G) \leq 4$ (The reader may note that if $\Delta(G) < 3$, then $a'(G) \leq 3$).*

Proof: We prove the Theorem by induction on the number of edges. The smallest possible number of edges in a non-regular connected graph G of maximum degree 3 on n vertices is $n-1$. Then clearly G is a tree and is acyclically edge colorable using 3 colors. Now let G be a non-regular connected graph on n vertices and $m \geq n$ edges with maximum degree 3. Let the Theorem be true for all non-regular connected graphs with maximum degree 3 with at most

$m-1$ edges. Without loss of generality we can assume that G is 2-connected, since if there are cut vertices in G, the acyclic edge coloring of the blocks of G can easily be extended to G. Thus $\delta(G) \geq 2$. Since G is not 3-regular, there is a vertex of degree 2. Let it be x.

Let $y \in N_G(x)$. Let $G' = G - \{xy\}$. Note that G' is connected, since G is 2-connected. If $\Delta(G') < 3$, then $a'(G') \leq 3$. Otherwise by induction hypothesis $a'(G') \leq 4$. Let $c : E' \rightarrow \{1,2,3,4\}$ be an acyclic edge coloring of G'. Let $F_y = \{c(y,z) | z \in N_{G'}(y)\}$. Note that $1 \leq |F_y| \leq 2$, since $2 \leq deg_G(y) \leq 3$. Let a' be the only neighbour of x, in G'. Let $S_{a'} = \{c(a',z) | z \in N_{G'}(a') - \{x\}\}$. Note that $|N_{G'}(y)| \geq 1$ and let $a \in N_{G'}(y)$. Let $S_{ya} = \{c(a,z) | z \in N_{G'}(a) - \{y\}\}$. Note that $1 \leq |S_{ya}| \leq 2$, since $2 \leq deg_{G'}(a) \leq 3$.

Our aim now is to extend the acyclic edge coloring c of G' to G by giving a color to the edge xy from the available 4 colors. Since $|F_y \cup \{c(x,a')\}| \leq 3$, there is at least one candidate color for the edge xy.

case 1: $c(x,a') \notin F_y$

Then clearly all the candidate colors are valid for the edge xy, since any cycle involving the edge xy will contain the edge xa' as well as an edge incident on y in G' and thus the cycle will have at least 3 colors.

case 2: $c(x,a') \in F_y$

Without loss of generality let $a \in N_{G'}(y)$ be the vertex such that $c(x,a') = c(y,a) = 1$. Suppose first $|N_{G'}(y)| = 1$. Then we have 3 candidate colors $\{2,3,4\}$. Suppose $\alpha \in \{2,3,4\}$ is not valid, what may be the reason? It is because if we assign color α to the edge xy, a bichromatic cycle is formed. It is easy to check that this has to be a (1,α) bichromatic cycle. It follows that if α is not valid there exists a (1,α) maximal bichromatic path with x and y as end vertices in G' with respect to the coloring c. Now if a color α is not valid, then it should be in $S_{a'}$ to form a (1,α) maximal bichromatic path. But $|S_{a'}| \leq 2$ and hence $|\{2,3,4\} - S_{a'}| \geq 1$. Thus at least one color from $\{2,3,4\}$ is $valid$ for the edge xy.

Now we can assume $|N_{G'}(y)| = 2$. Let $N_{G'}(y) = \{a,b\}$. Without loss of generality let $c(y,b) = 2$. Now colors $\{3,4\}$ are candidates for the edge xy. If both the colors 3 and 4 are not valid for the edge xy, then there are (1,3) and (1,4) maximal bichromatic paths starting at the vertex y, passing through vertex a and ending at vertex x. Thus $S_{a'} = \{3,4\}$ and $S_{ya} = \{3,4\}$.

Now recolor the edge xa' with color 2. Let the new coloring be called c'. Note that since $2 \notin S_{a'}$ and a is a pendant vertex in G', coloring c' is a valid acyclic edge coloring of the graph G'. Even with respect to the coloring c', colors $\{3,4\}$ are candidates for the edge xy. If both the colors 3 and 4 remain invalid for the edge xy even now, it means that there are (2,3) and (2,4) maximal bichromatic paths starting at the vertex y, passing through vertex b and ending at the vertex x. Thus $S_{yb} = \{3, 4\}$, where $S_{yb} = \{c(b, z) | z \in N_{G'}(b) - \{y\}\}$. Let P be the above discussed (2,4)- maximal bichromatic path with respect to c'. Note that P does not contain a as $2 \notin S_{ya}$.

Now we can exchange the colors of the edges ya and yb to get the coloring c''. That is, with respect to c'', we will have $c''(y, a) = 2$, $c''(y, b) = 1$ and $c''(e) = c'(e)$, for all other edges e in G'. Note that the coloring c'' is proper since $1 \notin S_{yb}$ and $2 \notin S_{ya}$. Now suppose there is a bichromatic cycle with respect to c''. Then clearly this bichromatic cycle should contain both the edges ya and yb as $deg_{G'}(y) = 2$. Moreover, there has to be an edge colored 2 at vertex b. Recall that $S_{yb} = \{3, 4\}$ and thus at vertex b only colors $\{1, 3, 4\}$ are present. Thus the coloring c'' is acyclic. What happens to the path P now? Since $c''(y, b) = 1$, the path P, which was bichromatic in coloring c' has 3 colors in the coloring c''. Let $P' = P - y$. It is easy to verify that P' is a (2,4) maximal bichromatic path which starts from vertex x and ends at vertex b and does not contain vertex a. We have $c''(y, a) = c''(x, a') = 2$. By fact 1 there can only be at most one (2,4)-maximal bichromatic path starting from the vertex x. We know P' is such a path and it does not include vertex a. Thus there can not be a (2,4) maximal bichromatic path which starts at vertex x and ends at vertex y, passing through vertex a. Thus color 4 is valid for the edge xy.

■

3.3 Comments

1. One natural question that may arise is whether the result can be extended to all subcubic graphs, i.e., is it true for all 3-regular graphs also? But this is in general not true since K_4 is a 3-regular graph which requires 5 colors. It was proved by Fiamcik [21], that every graph other than K_4 and $K_{3,3}$ is 4 colorable. But his paper was in Russian and was available only recently.

2. Every non-3-regular edge maximal connected graph with maximum degree 3 needs 4 colors to be properly acyclically edge colored. This is because, if n is even, then a matching can have at most $n/2$ edges. Only one color can take $n/2$ edges and all other colors can have a maximum of $n/2 - 1$ edges. Thus 3 colors can cover a maximum of $n/2 + 2(n/2 - 1) = 3n/2 - 2$ edges. But the edge maximal graph contains $3n/2 - 1$ edges

and thus needs 4 colors. If n is odd, then each matching can have maximum $(n-1)/2$ edges and all the three colors can cover at most $3(n-1)/2 = (3n-3)/2$ edges. But the edge maximal graph can have $(3n-1)/2$ edges and thus needs 4 colors. Thus our result is tight for all non-3-regular edge maximal connected subcubic graphs.

Chapter 4

Graphs with maximum degree 4

In this chapter we will see an upper bound for acyclic chromatic index for graphs with maximum degree 4.

4.1 Definitions and Preliminaries

Most of the definitions are as in the Chapter 2. Since the proof involves much more case analysis than other chapters on upper bounds, we give a more detailed notation for the operations $Recoloring$ and $Color\ Exchange$. This makes the presentation easier.

An obvious strategy to extend the coloring c of G' to G would be to try to assign one of the candidate colors in $C - F$ to the edge xy. The condition that a candidate color is not valid for the edge xy is captured in the following fact.

Fact 4.1. *The color $\beta \in C - F$ is not a $valid$ color for the edge xy if and only if $\exists \alpha \in F_x \cap F_y$ such that there is an (α, β, xy) critical path in G'.*

If none of the colors in $C - F$ is valid for the edge xy, then we can group the colors in $C - F$ into two categories namely $weak$ and $strong$.

Definition 4.2. ***Weak Color:*** *A color $\beta \in C - F$ is called $weak$ if it forms only one critical path with x and y as end points. Equivalently, there exists only one $\alpha \in F_x \cap F_y$ such that there is an (α, β, xy) critical path. Let $a \in N_{G'}(x)$. A weak color β is said to be* actively present *in a set S_{xa}, if $\exists k \in N_{G'}(a)$, such that $c(a, k) = \beta$ and the (α, β, xy) critical path contains the edge (a, k). Since $\beta \in C_F$ and $c(a, k) = \beta$, it is clear that $k \neq x$. If a weak color $\beta \in S_{xa}$ is* *not actively present in* S_{xa} *then it is said to be* passively present *in* S_{xa}.

Definition 4.3. ***Strong Color:****If the color* $\beta \in C - F$ *is not valid and also not* $weak$*, then it is called* $strong$*. Note that it appears on at least two critical paths.*

If there are weak colors, it makes sense to try to break the critical path containing one of the weak colors, thus enabling us to use that weak color for the edge xy. For this purpose we introduce the concept of $Recoloring$.

Definition 4.4. ***Recolor:*** *We define* $c' = Recolor(c, e, \gamma)$ *as the recoloring of the edge* e *with a candidate color* γ *to get a modified coloring* c' *from* c*, i.e.,* $c'(e) = \gamma$ *and* $c'(f) = c(f)$*, for all other edges* f *in* G'*. The recoloring is said to be proper, if the coloring* c' *is proper. The recoloring is said to be* $acyclic$ *(*$valid$*), if in coloring* c' *there exists no bichromatic cycle.*

Recall that our strategy is to extend the coloring of G' to G by assigning a valid color for the edge xy. When all the candidate colors of xy turn out to be $invalid$, we try to $slightly\ modify$ the coloring c of G' in such a way that with respect to the modified coloring, we have a valid color for edge xy. *Recoloring* of an edge in the critical path which contained a weak color is one such strategy. Sometimes we resort to a slightly more sophisticated strategy to modify the coloring namely $color\ exchange$ defined below.

Definition 4.5. ***Color Exchange:*** *Let* $u, i, j \in V(G')$ *and* $ui, uj \in E(G')$*. We define* $c' = ColorExchange(c, ui, uj)$ *as the the modification of the current coloring* c *by exchanging the colors of the edges* ui *and* uj*, i.e.,* $c'(u, i) = c(u, j)$*,* $c'(u, j) = c(u, i)$ *and* $c'(e) = c(e)$ *for all other edges* e *in* G'*. The color exchange with respect to the edges* ui *and* uj *is said to be proper if the coloring obtained after the exchange is proper. The color exchange with respect to the edges* ui *and* uj *is* $valid$ *if and only if the coloring obtained after the exchange is acyclic.*

In our proof we use the strategy of color exchange many times and in different contexts. All these contexts are more or less similar but differ in minor details. We would like to capture all these different contexts in a general framework. The configuration defined below is an attempt to formalize this:

Definition 4.6. ***Configuration A :*** *Let* u *be a vertex and* $i, j \in N_{G'}(u)$*. Let* $N'_{G'}(u) \cup N''_{G'}(u)$ *be a partition of* $N_{G'}(u) - \{i, j\}$*, i.e.,*$N'_{G'}(u) \cup N''_{G'}(u) = N_{G'}(u) - \{i, j\}$ *and* $N'_{G'}(u) \cap N''_{G'}(u) = \emptyset$*. The 5-tuple* $(u, i, j, N'_{G'}(u), N''_{G'}(u))$ *is in* $configuration\ A$ *if*

1. $c(u,i) \notin S_{uj}$ *and* $c(u,j) \notin S_{ui}$

2. $\forall z \in N'_{G'}(u)$, $c(u,z) \notin S_{ui}$ *and* $c(u,z) \notin S_{uj}$

Suppose $(u,i,j,N'_{G'}(u),N''_{G'}(u))$ is in $configuration\ A$ with respect to the coloring c. Let c' be the coloring obtained after the color exchange with respect to the edges ui and uj. Then note that condition 1 guarantees that the color $c(u,i)$ is a candidate for edge uj and the color $c(u,j)$ is a candidate for edge ui and thus the coloring obtained after the color exchange is proper. Condition 2 inhibits the possibility of any $(c(u,i),c(u,z))$ or $(c(u,j),c(u,z))$ $bichromatic\ cycles$ being formed for any $z \in N'_{G'}(u)$. Its obvious that there can not be any $(c(u,j),c(u,i))$ $bichromatic\ cycles$ after exchange. Thus the following fact is easy to verify:

Fact 4.7. *Let the 5-tuple* $(u,i,j,N'_{G'}(u),N''_{G'}(u))$ *be in* $configuration\ A$. *Then the operation* $c' = ColorExchange(c,ui,uj)$ *is not* $valid$ *if and only if* $\exists h \in N''_{G'}(u)$ *such that after the color exchange (i.e., in* c'*) there exists an* (α,β) *bichromatic cycle that passes through* h *for* $\alpha \in \{c'(u,i),c'(u,j)\}$ *and* $\beta = c'(u,h)$.

In view of $Fact$ 4.7, the following $Fact$ is obvious:

Fact 4.8. *Let the 5-tuple* $(u,i,j,N'_{G'}(u),N''_{G'}(u))$ *be in* $configuration\ A$. *Then if* $N''_{G'}(u) = \emptyset$, *the color exchange* $c' = ColorExchange(c,ui,uj)$ *is* $valid$.

4.2 The Theorem

Theorem 4.9. *Let* G *be a connected graph on* n *vertices,* $m \leq 2n-1$ *edges and maximum degree* $\Delta \leq 4$, *then* $a'(G) \leq 6$. (*Note that if* $\Delta(G) \leq 4$, *then* $m \leq 2n$ *always*).

Proof: We prove the Theorem by induction on the number of edges. Let $H = (V_H, E_H)$ be a connected graph of n vertices and $m \leq 2n-1$ edges and $\Delta(H) \leq 4$. Trivially the Theorem is true for $|E| = m = 0$. Let the Theorem be true for all connected graphs W such that $\Delta(W) \leq 4$ and $|E(W)| \leq 2|V(W)|-1$, with at most $m-1$ edges. Without loss of generality we can assume that H is 2-connected, since if there are cut vertices in H, the acyclic edge coloring of the blocks $B_1, B_2 \ldots B_k$ of H can easily be extended to H (Note that each block satisfies the property that $\Delta(B_i) \leq 4$ and $|E(B_i)| \leq 2|V(B_i)|-1$). Thus $\delta(H) \geq 2$ ($\delta(H)$ denotes the minimum degree of graph H). Now since H has at most $2n-1$ edges, there is a vertex x of degree at most 3.

Let $y \in N_H(x)$. The degree of y is at most 4. Let $H' = H - \{xy\}$, i.e.,$H' = (V_{H'}, E_{H'})$, where $V_{H'} = V_H$ and $E_{H'} = E_H - \{xy\}$. Thus in H', $deg(x) \leq 2$ and $deg(y) \leq 3$. Note that since H is 2-connected, H' is connected.

To avoid certain technicalities in the presentation of the proof, we construct the graph G' from H' as below. If $deg_{H'}(x) = 2$, $deg_{H'}(y) = 3$ and $\forall z \in N_{H'}(x) \cup N_{H'}(y), deg_{H'}(z) = 4$, then let $G' = H'$ and $G = H$. Otherwise, we construct the graph $G' = (V', E')$ from H' in the following manner. First add pendant vertices as neighbours to the vertices x and y such that $deg_{G'}(x) = 2$ and $deg_{G'}(y) = 3$. Next add pendant vertices as neighbours to the newly added vertices and $\forall z \in N_{H'}(x) \cup N_{H'}(y)$ such that $\forall z \in N_{G'}(x) \cup N_{G'}(y), deg_{G'}(z) = 4$. Note that since H' was connected, G' is also connected. Let $G = G' \cup \{xy\}$, i.e., $G = (V, E)$, where $V = V'$ and $E = E' + \{xy\}$.

By induction hypothesis, graph H' is acyclically edge colorable using 6 colors. Note that we can easily extend the coloring of H' to G' by coloring each of the newly added edges with the available colors satisfying the acyclic edge coloring property. Let $c_0 : E' \rightarrow \{1, 2, ..., 6\}$ be an acyclic edge coloring of G'. It is easy to see that if we extend the acyclic edge coloring of G' to G by assigning an appropriate color to the edge xy, then this coloring also corresponds to the acyclic edge coloring of H, since H is a subgraph of G.

Our intention will be to extend the acyclic edge coloring c_0 of G' to $G = G' + \{xy\}$ by assigning an appropriate color for the edge xy. We denote the set of colors of c_0 by $C = \{1, 2, 3, 4, 5, 6\}$.

Let $N_{G'}(x) = \{a, b\}$ and $N_{G'}(y) = \{a', b', d'\}$. Note that $N_{G'}(x) \cap N_{G'}(y)$ need not be empty. Also recall that $deg_{G'}(a) = deg_{G'}(b) = 4$. Let $N_{G'}(a) = \{x, k_1, k_2, k_3\}$ and $N_{G'}(b) = \{x, l_1, l_2, l_3\}$.

case 1: $F_x \cap F_y = \emptyset$

Since $|F| = 5$, $|C - F| = 1$. Clearly the $candidate$ color in $C - F$ is valid for the edge xy.

case 2: $|F_x \cap F_y| = 2$

Assumption 4.10. *Without loss of generality let* $F_x = \{1, 2\}$ *and* $F_y = \{1, 2, 3\}$. *Thus* $F = \{1, 2, 3\}$.

By $Assumption$ 4.10, $C - F = \{4, 5, 6\}$. If none of the candidate colors are $valid$, then by $Fact$ 4.1, the following Claim is easy to see:

Claim 4.11. *With respect to the coloring* c_0, $\forall \beta \in C - F, \exists \alpha \in F_x \cap F_y$ *such that there is a* (α, β, xy) *critical path.*

case 2.1:$(S_{xa} \cup S_{xb}) \cap F = \emptyset$

Since $F = \{1, 2, 3\}$, $S_{xa} = S_{xb} = \{4, 5, 6\}$.

Claim 4.12. *With respect to the coloring c_0, all the colors of $C - F$ are weak.*

Proof: Suppose not. Then there is a strong color in $C - F$. Without loss of generality let 4 be a strong color. Let $c_0(x, a) = c_0(y, a') = 1$ and $c_0(x, b) = c_0(y, b') = 2$. Now it is easy to check that the 5-tuple $(x, a, b, \emptyset, \emptyset)$ satisfies $configuration\ A$. Let

$$c'_0 = ColorExchange(c_0, xa, xb)$$

By $Fact$ 4.8 the color exchange with respect to the edges xa and xb is valid. Thus the coloring c'_0 is acyclic.

Since color 4 was strong in coloring c_0, there was a $(1, 4, xy)$ critical path as well as a $(2, 4, xy)$ critical path before $color\ exchange$ (i.e., with respect to the coloring c_0). Thus by $Lemma$ 2.8, $(1, 4, xy)$ critical path and $(2, 4, xy)$ critical path will not exist after the $color\ exchange$ (i.e., with respect to the coloring c'_0). Thus by $Fact$ 4.1, color 4 is valid for edge xy. Thus we infer that with respect to the coloring c_0, all the colors of $C - F$ are weak. □

By $Claim$ 4.12, all the colors of $C - F$ are weak. Each weak color should be actively present in exactly one of S_{xa} or S_{xb}. Since there are 3 weak colors, we can infer that either S_{xa} or S_{xb} is such that at least 2 of the weak colors are actively present in it.

Assumption 4.13. *Without loss of generality assume that colors 4 and 5 are $actively\ present$ in S_{xa}. Let $c_0(a, k_1) = 4$ and $c_0(a, k_2) = 5$.*

From $Assumption$ 4.13, it follows that since $c_0(x, a) = 1$, there exist $(1, 4, xy)$ and $(1, 5, xy)$ *critical paths*. The following claim is obvious.

Claim 4.14. *With respect to the coloring c_0, $1 \in S_{ak_1}$ and $1 \in S_{ak_2}$.*

It is easy to verify that the 5-tuple $(x, a, b, \emptyset, \emptyset)$ satisfies configuration A with respect to the coloring c_0.

$$c_1 = ColorExchange(c_0, xa, xb)$$

By $Fact$ 4.8 the color exchange with respect to the edges xa and xb is valid. Thus the coloring c_1 is acyclic.

But there were $(1, 4, xy)$ and $(1, 5, xy)$ *critical paths* before $color\ exchange$ (i.e., with respect to the coloring c_0). By $Lemma$ 2.8, both $(1, 4, xy)$ and $(1, 5, xy)$ *critical paths* do not exist after the $color\ exchange$ (i.e., with respect to the coloring c_1).

Thus even with respect to the coloring c_1, if both the colors 4 and 5 are not $valid$ for the edge xy, by $Fact$ 4.1, there has to be $(2, 4, xy)$ and $(2, 5, xy)$ *critical paths*. Thus $2 \in S_{ak_1}$ and $2 \in S_{ak_2}$. Thus we can $Claim$ the following:

Claim 4.15. *With respect to the coloring* c_1, $\{1, 2\} \subset S_{ak_1}$ *and* $\{1, 2\} \subset S_{ak_2}$. *Moreover there will not be any* $(1, 4, xy)$ *and* $(1, 5, xy)$ critical paths.

Now since the colors 4 and 5 are weak, we try to break the $(2, 4, xy)$ and $(2, 5, xy)$ *critical paths* by recoloring the edge xa.

$$c_2 = Recolor(c_1, xa, 3)$$

Note that color 3 is a candidate for the edge xa since $S_{xa} = \{4, 5, 6\}$ and $c_1(x, b) = 1$. And also since $S_{xa} \cap S_{ax} = \emptyset$, by $Lemma$ 2.3 color 3 is $valid$ for the edge xa.

Note that with respect to the coloring c_2, $F_x \cap F_y = \{1, 3\}$. In view of $Claim$ 4.15, there will not be any $(1, 4, xy)$ and $(1, 5, xy)$ *critical paths* with respect to the coloring c_2 also. If both the colors 4 and 5 are not $valid$ for the edge xy still, then by $Fact$ 4.1, there has to be $(3, 4, xy)$ and $(3, 5, xy)$ *critical paths* implying $3 \in S_{ak_1}$ and $3 \in S_{ak_2}$. Thus combined with $Claim$ 4.15, we infer the following:

Claim 4.16. *With respect to the coloring* c_2, *we have* $S_{ak_1} = S_{ak_2} = \{1, 2, 3\}$. *Moreover there will not be any* $(1, 4, xy)$ *and* $(1, 5, xy)$ critical paths.

Now the 5-tuple $(a, k_1, k_2, \{k_3\}, \{x\})$ satisfies configuration A.

$$c_3 = ColorExchange(c_2, ak_1, ak_2)$$

By fact 4.7 if there is any bichromatic cycle (recalling that $c_3(a, x) = 3$), it has to be either a $(5, 3)$ or $(6, 3)$ bichromatic cycle that passes through vertex a and hence vertex x. But any cycle that passes through vertex x should contain edge xb also. Since $c_3(x, b) = 1$, this is a contradiction and we infer that c_3 is acyclic.

There was a $(3, 4, xy)$ critical path as well as a $(3, 5, xy)$ critical path before $color\ exchange$ (i.e., with respect to the coloring c_2). Thus by $Lemma$ 2.8, both these critical paths does not exist after the color exchange (i.e., with respect to the coloring c_3) (Note that k_1, $k_2 \notin \{x, y\}$ since $c_2(a, k_1) = 4$ and $c_2(a, k_2) = 5 \notin F_x$ or F_y. Therefore we can apply $Lemma$ 2.8)

To summarize, $c_3(x, a) = 3$, $c_3(x, b) = 1$ and thus $F_x \cap F_y = \{1, 3\}$. With respect to the coloring c_3, there exist no $(3, 4, xy)$ and $(3, 5, xy)$ critical paths. Recall that by Claim 4.16, there won't be any $(1, 4, xy)$ and $(1, 5, xy)$ critical paths with respect to the coloring c_2. It is easy to see that even with respect to the coloring c_3, there won't be any $(1, 4, xy)$ and $(1, 5, xy)$ critical paths.

Thus by $Fact$ 4.1, color 4 and 5 are valid for edge xy.

case 2.2:$(S_{xa} \cup S_{xb}) \cap F \neq \emptyset$

Assumption 4.17. *Without loss of generality let $S_{xa} \cap F \neq \emptyset$. It follows that one of $\{4, 5, 6\}$ is missing in S_{xa} since $|S_{xa}| = 3$. Without loss of generality let it be color 5. Also let $c_0(x, a) = c_0(y, a') = 1$ and $c_0(x, b) = c_0(y, b') = 2$ and $c_0(y, d') = 3$.*

Claim 4.18. *With respect to the coloring c_0, there exists a $(2, 5, xy)$ $critical\ path$. Thus $5 \in S_{xb}$.*

Proof: Since color 5 is not valid for the edge xy, by $Claim$ 4.11 there has to be a $(1, 5, xy)$ critical path or a $(2, 5, xy)$ critical path. But by $Assumption$ 4.17, color $5 \notin S_{xa}$ and hence there can not be a $(1, 5, xy)$ critical path. Thus there exists a $(2, 5, xy)$ critical path. ☐

Claim 4.19. *With respect to the coloring c_0, all the colors of $C - F$ are weak.*

Proof: Suppose not. Then there is at least one strong color in $C - F$. Without loss of generality let 4 be a strong color. Thus we have $4 \in S_{xb}$. Combined with $Claim$ 4.18, we have:

$$\{4, 5\} \subset S_{xb}. \tag{4.1}$$

Now let

$$c_0' = Recolor(c_0, xa, 5)$$

Note that color 5 is a candidate for the edge xa since $c_0(x, b) = 2$ and $5 \notin S_{xa}$ (by $Assumption$ 4.17). Now we claim that assigning color 5 to the edge xa can not result in any bichromatic cycle. To see this first note that since any cycle containing the edge xa should also contain the edge xb, but $c_0(x, b) = 2$ and therefore if a bichromatic cycle gets formed it must be a $(2, 5)$ bichromatic cycle, implying that there is a $(2, 5, xa)$ critical path. But there is already a $(2, 5, xy)$ critical path (by $Claim$ 4.18) and by $Fact$ 2.1 there can not be a $(2, 5, xa)$ critical path, a contradiction. Thus coloring c_0' is acyclic.

Note that with respect to the coloring c_0', color 6 remains to be a candidate color for the edge xy. Also note that $F_x \cap F_y = \{2\}$. If the candidate color 6 is not valid for the edge xy, then by $Fact$ 4.1 there has to be a $(2, 6, xy)$ critical path and thus $6 \in S_{xb}$. Thus combined with (4.1), we have:

$$S_{xb} = \{4, 5, 6\} \tag{4.2}$$

With respect to the coloring c_0, color 4 was strong (assumption) and thus there existed a $(1, 4, xy)$ critical path. After recoloring the edge xa with color 5 (i.e., with respect to the coloring c'_0), the $(1, 4, xy)$ critical path gets curtailed to a $(1, 4, y, a)$ maximal bichromatic path without containing the vertex x. Moreover note that $(1, 4, y, a)$ maximal bichromatic path does not contain the vertex b, since if b is in this path, then it is an internal vertex and thus both colors $1, 4 \in F_b$, a contradiction ($1 \notin F_b$). Thus we have,

With respect to the coloring c'_0, a $(1, 4, y, a)$ maximal bichromatic path exists, but this path does not contain the vertices x or b. (4.3)

Now with respect to the coloring c'_0, $F_x \cap F_y = \{2\}$. Let

$$c''_0 = Recolor(c'_0, xb, 1)$$

Note that color 1 is a candidate color for the edge xb since $c'_0(x, a) = 5$ and $1 \notin S_{xb} = \{4, 5, 6\}$. Color 1 is $valid$ for the edge xb because any bichromatic cycle containing edge xb should also contain edge xa and since color $1 \notin S_{xa}$ (Recall that $c_0(x, a) = 1$. Thus $1 \notin S_{xa}$ with respect to the coloring c_0. Therefore $1 \notin S_{xa}$ with respect to the coloring c'_0 also.), such a $(1, 5)$ bichromatic cycle can not be formed. Thus c''_0 is acyclic.

Thus with respect to coloring c''_0, $F_x \cap F_y = \{1\}$. Now by (4.3), with respect to the coloring c'_0, there existed a $(1, 4, y, a)$ maximal bichromatic path that does not contain vertex b or x. Thus noting that c''_0 is obtained just by changing the color of the edge xb to 1, by $Fact$ 2.1 we infer that c''_0 can not contain $(1, 4, xy)$ critical path.

Thus by $Fact$ 4.1 color 4 is valid for the edge xy. Thus we can infer that with respect to the coloring c_0, all the colors of $C - F$ are weak. □

Claim 4.20. *In view of $Assumption$ 4.17, with respect to the coloring c_0, each $\alpha \in \{4, 5, 6\}$ is $actively\ present$ in S_{xb}.*

Proof: Suppose not. By $Claim$ 4.18, we know that color 5 is $actively\ present$ in S_{xb}. Without loss of generality let color 6 be not $actively\ present$ in S_{xb}. Therefore color 6 is

$actively\ present$ in S_{xa}. Now let

$$c_0' = Recolor(c_0, xa, 5)$$

Note that color 5 is a candidate since $5 \notin S_{xa}$ (by $Assumption$ 4.17) and $c_0(x,b) = 2$. Now we claim that assigning color 5 to the edge xa can not result in any bichromatic cycle. To see this first note that since any cycle containing the edge xa should also contain the edge xb, but $c_0(x,b) = 2$ and therefore if a bichromatic cycle gets formed it must be a $(2,5)$ bichromatic cycle, implying that there is a $(2,5,xa)$ critical path with respect to the coloring c_0. But in c_0 there is already a $(2,5,xy)$ critical path (by Claim 4.11 and Claim 4.18) and by $Fact$ 2.1 there can not be a $(2,5,xa)$ critical path, a contradiction. Thus coloring c_0' is acyclic.

Now $F_x \cap F_y = \{2\}$. But in c_0, there did not exist a $(2,6,xy)$ critical path since by assumption color 6 is not actively present in S_{xb}. Thus noting that c_0' is obtained just by changing the color of the edge xa to 5, we infer that c_0' can not contain $(2,6,xy)$ critical path.

Thus by $Fact$ 4.1 color 6 is valid for the edge xy. We infer that with respect to the coloring c_0, each $\alpha \in \{4,5,6\}$ is $actively\ present$ in S_{xb}. □

Recall that $c_0(x,b) = c_0(y,b') = 2$. In view of $Claim$ 4.20, with respect to the coloring c_0, we have:

$$S_{xb} = S_{yb'} = \{4,5,6\} \tag{4.4}$$

Let

$$c_1 = Recolor(c_0, xb, 3)$$

Note that color 3 is a candidate for edge xb since $3 \notin S_{xb} = \{4,5,6\}$ (by (4.4)) and $c_0(x,a) = 1$. Moreover since $S_{xb} \cap S_{bx} = \emptyset$, by $Lemma$ 2.3 color 3 is also $valid$. Thus the coloring c_1 is acyclic.

With respect to the coloring c_1, $F_x \cap F_y = \{1,3\}$. In view of $Claim$ 4.19 and $Claim$ 4.20, $\forall \alpha \in \{4,5,6\}$, α is not $actively\ present$ in S_{xa} and thus $(1,\alpha,xy)$ critical path does not exist with respect to the coloring c_0. It is true with respect to the coloring c_1 also. Hence if none of the colors from $\{4,5,6\}$ is $valid$ for the edge xy with respect to the coloring c_1, then by $Fact$ 4.1 there has to be $(3,4,xy)$, $(3,5,xy)$ and $(3,6,xy)$ *critical paths*. Recalling that by $Assumption$ 4.17 $c(y,d') = 3$, we infer that $S_{yd'} = \{4,5,6\}$.

Thus with respect to the coloring c_1, we have:

$$S_{yb'} = S_{yd'} = \{4, 5, 6\} \tag{4.5}$$

The 5-tuple $(y, b', d', \{a'\}, \emptyset)$ is configuration A. Now let

$$c_2 = ColorExchange(c_1, yb', yd')$$

By $Fact$ 4.8 the color exchange with respect to the edges yb' and yd' is valid. Thus the coloring c_2 is acyclic.

For $\alpha \in \{4, 5, 6\}$ there was a $(3, \alpha, xy)$ critical path before $color\ exchange$ (with respect to coloring c_1). Thus by $lemma$ 2.8, these critical paths do not exist after the $color\ exchange$ (with respect to coloring c_2). Also recall that there was no $(1, \alpha, xy)$ critical path, for $\alpha \in \{4, 5, 6\}$, with respect to the coloring c_1. Noting that the $color\ exchange$ involved only the colors 2 and 3 there is no chance of any $(1, \alpha, xy)$ critical path to get formed with respect to the coloring c_2.

Thus by $fact$ 4.1, color α is valid for edge xy.

case 3: $|F_x \cap F_y| = 1$

Assumption 4.21. *Without loss of generality let* $F_x = \{1, 2\}$ *and* $F_y = \{1, 3, 4\}$. *Thus* $F = \{1, 2, 3, 4\}$. *Then* $C - F = \{5, 6\}$. *Let* $c_0(x, a) = c_0(y, a') = 1$, $c_0(x, b) = 2$, $c_0(y, b') = 3$ *and* $c_0(y, d') = 4$.

If none of the colors from $C - F$ are $valid$, then by $Fact$ 4.1, there exist $(1, 5, xy)$ and $(1, 6, xy)$ $critical\ paths$. We capture this in the following $claim$:

Claim 4.22. *With respect to coloring* c_0, *there exist* $(1, 5, xy)$ *and* $(1, 6, xy)$ $critical\ paths$. *Thus* $\{5, 6\} \subset S_{xa}$ *and* $\{5, 6\} \subset S_{ya'}$.

Claim 4.23. *With respect to coloring* c_0, $\{3, 4\} \subset S_{xb}$.

Proof: Suppose not. Then at least one of 3, 4 is missing in S_{xb}. Without loss of generality let $4 \notin S_{xb}$. Recalling that $c_0(x, a) = 1$, it follows that color 4 is a candidate color for the edge xb. We claim that there exists a $(1, 4, xb)$ critical path with respect to the coloring c_0. Suppose not. Then let

$$c'_0 = Recolor(c_0, xb, 4)$$

Clearly c'_0 is acyclic since any bichromatic cycle being formed should involve the edge xa

as well. But $c_0'(x,a) = 1$ and hence a $(1,4)$ bichromatic cycle has to be formed, implying that there is a $(1,4,xb)$ critical path, a contradiction to our assumption.

With respect to the coloring c_0', $|(F_x \cap F_y) = \{1,4\}| = 2$, and by $case$ 2 we will be able to find a valid color for the edge xy.

Thus we can infer that there exists a $(1,4,xb)$ critical path with respect to the coloring c_0. For a $(1,4,xb)$ critical path to exist clearly we should have $4 \in S_{xa}$, since $c_0(x,a) = 1$. Combined with $Claim$ 4.22, we get:

$$S_{xa} = \{4,5,6\} \tag{4.6}$$

Moreover we have $1 \in S_{xb}$ with respect to c_0 since there is a $(1,4,xb)$ critical path. Now let the other two colors in S_{xb} be $\{\alpha,\beta\}$. Then $\gamma \in (\{3,5,6\} - \{\alpha,\beta\})$ is a candidate color for the edge xb. Let

$$c_0'' = Recolor(c_0, xb, \gamma)$$

We claim that c_0'' is acyclic. Otherwise if any bichromatic cycle gets formed with respect to the coloring c_0'', then it should be a $(\gamma,1)$ bichromatic cycle since any cycle that contains edge xb should contain edge xa also and $c_0''(x,a) = 1$, implying that there exists a $(1,\gamma,xb)$ critical path with respect to the coloring c_0. If $\gamma = 3$, such a critical path can not exist since $3 \notin S_{xa}$ (by (4.6)). On the other hand if $\gamma \in \{5,6\}$, by $Fact$ 2.1, $(1,\gamma,xb)$ critical path can not exist with respect to the coloring c_0 since there is already a $(1,\gamma,xy)$ critical path (by $Claim$ 4.22). Thus we infer that c_0'' is acyclic.

With respect to coloring c_0'', if $\gamma = 3$, $|(F_x \cap F_y) = \{1,3\}| = 2$, and by $case$ 2 we will be able to find a valid color for the edge xy.

With respect to coloring c_0'', if $\gamma \in \{5,6\}$ we have $(F_x \cap F_y) = \{1\}$ and $2 \in C - F$. Thus color 2 is a candidate color for the edge xy. Moreover since $S_{xa} = \{4,5,6\}$ (by (6)), there can not be a $(1,2,xy)$ critical path and hence by $Fact$ 4.1, color 2 is valid for the edge xy. Thus we infer that with respect to coloring c_0, $\{3,4\} \subset S_{xb}$. □

Claim 4.24. *With respect to the coloring* c_0, $S_{xb} = \{3,4,1\}$.

Proof: Suppose not. Then in view of Claim 4.23, we can infer that color $1 \notin S_{xb}$. Recall that by Claim 4.22, $\{5,6\} \subset S_{xa}$. Let the remaining color in S_{xa} be α. Let $\beta \in \{3,4\} - \{\alpha\}$. Now let

$$c'_0 = Recolor(c_0, xb, 1)$$

and

$$c''_0 = Recolor(c'_0, xa, \beta)$$

Note that c''_0 is proper since $1 \notin S_{xb}$ (by $Assumption$) and $\beta \notin S_{xa}$, by the definition of β. The coloring c''_0 is acyclic since any cycle containing the edge xa should also contain the edge xb (and vise versa), but $c''_0(x, b) = 1$ and therefore if a bichromatic cycle gets formed it must be a $(1, \beta)$ bichromatic cycle, implying that $1 \in S_{xa}$. But this is a contradiction since $1 \notin S_{xa}$ with respect to c_0 as $c_0(x, a) = 1$ and therefore $1 \notin S_{xa}$ with respect to c''_0 also.

Now since $\beta \in \{3, 4\}$, we have $|(F_x \cap F_y) = \{1, \beta\}| = 2$ and thus the situation reduces to $case$ 2, thereby enabling us to find a valid color for the edge xy. Thus we infer that with respect to the coloring c_0, $S_{xb} = \{3, 4, 1\}$. □

Claim 4.25. *There is a* $(1, 2, xy)$ *critical path . Thus in combination with* $Claim$ *4.22* $S_{xa} = \{5, 6, 2\}$ *,* $S_{ya'} = \{5, 6, 2\}$ *with respect to the coloring* c_0*.*
Proof: Suppose not. Let

$$c'_0 = Recolor(c_0, xb, 5)$$

Note that color 5 is a candidate color for the edge xb since, by $Claim$ 4.24, $S_{xb} = \{3, 4, 1\}$ and $c_0(x, a) = 1$. It is also valid since if there is a bichromatic cycle, then it should contain the edges xa and xb and hence it has to be a $(1, 5)$ bichromatic cycle, implying that there exists a $(1, 5, xb)$ critical path with respect to the coloring c_0. But there can not be a $(1, 5, xb)$ critical path (by $Fact$ 2.1) as there is already a $(1, 5, xy)$ critical path (by $Claim$ 4.22). Thus the coloring c'_0 is acyclic.

Now with respect to the coloring c'_0, $F_x \cap F_y = \{1\}$. Color 2 is a candidate color for the edge xy since $2 \notin (F_x \cup F_y = \{1, 3, 4, 5\})$. Since there is no $(1, 2, xy)$ critical path (by assumption), by $Fact$ 4.1, color 2 is valid for the edge xy. Thus we can infer that there exists a $(1, 2, xy)$ critical path with respect to the coloring c_0. □

Recall that $N_{G'}(a) = \{x, k_1, k_2, k_3\}$ and $N_{G'}(b) = \{x, l_1, l_2, l_3\}$. Also recall that by Assumption 4.21, $c_0(x, a) = c_0(y, a') = 1$,$c_0(x, b) = 2$,$c_0(y, b') = 3$ and $c_0(y, d') = 4$. By

Claim 4.24 and Claim 4.25, $S_{xa} = \{5,6,2\}$ and $S_{xb} = \{3,4,1\}$. We make the following $Assumption$:

Assumption 4.26. *Without loss of generality let* $c_0(a,k_1) = 5$, $c_0(a,k_2) = 6$, $c_0(a,k_3) = 2$, $c_0(b,l_1) = 3$, $c_0(b,l_2) = 4$ *and* $c_0(b,l_3) = 1$.

The main intention of the next two $Claims$ is to establish that $S_{bl_1} = S_{bl_2} = \{2,5,6\}$.

Claim 4.27. *With respect to the coloring* c_0*, there exist* $(2,3,xa)$ *and* $(2,4,xa)$ *critical paths. Thus* $2 \in S_{bl_1}$, $2 \in S_{bl_2}$.

Proof: Suppose not. Then without loss of generality let there be no $(2,3,xa)$ critical path. Let

$$c_0' = Recolor(c_0, xa, 3)$$

Note that color 3 is a candidate color for edge xa since $3 \notin S_{xa} = \{2,5,6\}$ (by Claim 4.25) and $c_0(x,b) = 2$. It is also valid since if there is any bichromatic cycle containing edge xa, then it should also contain edge xb and since $c_0(x,b) = 2$, it has to be a $(2,3)$ bichromatic cycle, implying that there is a $(2,3,xa)$ critical path, a contradiction to our assumption. Thus the coloring c_0' is acyclic.

With respect to the coloring c_0', $c_0'(y,b') = 3$ and $(F_x \cap F_y) = \{3\}$. Now if one of the colors 5 and 6 are valid for the edge xy, we are done. Otherwise by $Fact$ 4.1, there are $(3,5,xy)$ and $(3,6,xy)$ critical paths. Thus

$$\{5,6\} \subset S_{yb'} \tag{4.7}$$

Let,

$$c_0'' = Recolor(c_0', xb, 5)$$

First note that color 5 is a candidate for the edge xb since $5 \notin S_{xb} = \{3,4,1\}$ (by $Claim$ 4.24) and $c_0'(x,a) = 3$. It is also valid since if there is any bichromatic cycle containing the edge xb then it should also contain edge xa and since $c_0'(x,a) = 3$, it has to be a $(3,5)$ bichromatic cycle,implying that there exists a $(3,5,xb)$ critical path. But there can not be a $(3,5,xb)$ critical path (by $Fact$ 2.1) as there is already a $(3,5,xy)$ critical path. Thus the coloring c_0'' acyclic.

Now with respect to the coloring c_0'', $(F_x \cap F_y) = \{3\}$ and $2 \notin (F_x \cup F_y) = \{1,3,4,5\}$. Color 2 is a $candidate$ for the edge xy. If it is $valid$ then we are done. Otherwise by $Fact$ 4.1, there exists a $(3,2,xy)$ critical path.

Thus $2 \in S_{yb'}$ and in combination with (4.7), we get,

$$S_{yb'} = \{2, 5, 6\} \tag{4.8}$$

Recall that $S_{ya'} = \{2, 5, 6\}$ by $Claim$ 4.25 with respect to the coloring c_0. It is easy to see that $S_{ya'} = \{2, 5, 6\}$ even with respect to the coloring c_2. Now in view of Assumption 4.21, we have the 5-tuple $(y, a', b', \{d'\}, \emptyset)$ in $Configuration\ A$. Let,

$$c_0''' = ColorExchange(c_0'', ya', yb')$$

By $Fact$ 4.8, the color exchange with respect to the edges ya' and yb' is valid. Thus the coloring c_0''' is acyclic.

There was a $(3, 6, xy)$ critical path before $color\ exchange$ (i.e., with respect to the coloring c_0'') since otherwise color 6 would have been valid for the edge xy with respect to the coloring c_0''. Thus by $Lemma$ 2.8 no $(3, 6, xy)$ critical path exists after the $color\ exchange$ (i.e., with respect to the coloring c_0'''). Thus by $Fact$ 4.1, color 6 is valid for edge xy. We can infer that with respect to the coloring c_0, there exist $(2, 3, xa)$ and $(2, 4, xa)$ critical paths. □

Claim 4.28. *With respect to the coloring* c_0, $\forall\alpha \in \{3, 4\}$ *and* $\forall\beta \in \{5, 6\}$, *there exist* (α, β, b, a) *maximal bichromatic path which ends at vertex* a *with an edge colored* β. *Thus* $S_{bl_1} = \{2, 5, 6\}$ *and* $S_{bl_2} = \{2, 5, 6\}$.

Proof: Suppose not. Then $\exists\alpha \in \{3, 4\}$ and $\exists\beta \in \{5, 6\}$ such that there is no (α, β, b, a) maximal bichromatic path which ends at vertex a with an edge colored β. Without loss of generality let $\alpha = 3$ and $\beta = 5$. Now let,

$$c_0' = Recolor(c_0, xa, 3)$$

and

$$c_0'' = Recolor(c_0', xb, 5)$$

Note that c_0'' is a proper coloring (since ($3 \notin S_{xa} = \{2, 5, 6\}$ and $c_0''(x, b) = 5$) and ($5 \notin S_{xb} = \{3, 4, 1\}$ and $c_0''(x, a) = 3$)). Now to see that c_0'' is acyclic, note that if there is a bichromatic cycle with respect to the coloring c_0'', then it should contain both the edges xa and xb, thus forming $(3, 5)$ bichromatic cycle, implying that there should be a $(3, 5, a, b)$ maximal bichromatic path which ends at vertex a with an edge colored 3 with respect to the coloring c_0, a contradiction to our assumption.

Note that with respect to the coloring c_0'', $F = \{1, 3, 4, 5\}$ and thus color 2 is a candidate color for the edge xy. By $Claim$ 4.27 there was a $(2, 3, xa)$ critical path with respect to the coloring c_0. From this it is easy to see that with respect to the coloring c_0'', there is a $(3, 2, xb)$ critical path. Thus by $Fact$ 2.1 there can not be a $(3, 2, xy)$ critical path with respect to the coloring c_0''. Hence color 2 is valid for the edge xy.

Thus $\forall \alpha \in \{3, 4\}$ and $\forall \beta \in \{5, 6\}$, there exist (α, β, b, a) maximal bichromatic path which ends at vertex a with an edge colored β. Thus recalling that $c_0(b, l_1) = 3$ and $c_0(b, l_2) = 4$ with respect to the coloring c_0, we have,

$$\{5, 6\} \subset S_{bl_1} \tag{4.9}$$

$$\{5, 6\} \subset S_{bl_2} \tag{4.10}$$

By $Claim$ 4.27, $2 \in S_{bl_1}$ and $2 \in S_{bl_2}$. Thus we have,

$$S_{bl_1} = S_{bl_2} = \{2, 5, 6\} \tag{4.11}$$

□

Now let,

$$c_1 = Recolor(c_0, xb, 5)$$

Recalling Claim 4.24, $s_{xb} = \{3, 4, 1\}$ and $c_0(x, a) = 1$, color 5 is a candidate for the edge xb. Moreover color 5 is also valid since if there is any bichromatic cycle containing the edge xb then it should also contain edge xa and since $c_0(x, a) = 1$, it has to be a $(1, 5)$ bichromatic cycle,implying that there exists a $(1, 5, xb)$ critical path with respect to the coloring c_0. But there can not be a $(1, 5, xb)$ critical path (by $Fact$ 2.1) as there is already a $(1, 5, xy)$ critical path (by $Claim$ 4.22). Thus the coloring c_1 is acyclic.

Recall that by $Claim$ 4.28, with respect to the coloring c_0, there was a $(3, 5, b, a)$ maximal bichromatic path that ends at vertex a with an edge colored 5. After the recoloring of edge xb with color 5 (i.e., with respect to the coloring c_1),it is easy to see that this $(3, 5, b, a)$ maximal bichromatic path gets extended to a $(5, 3, xa)$ critical path. Thus we have,

$$\textit{With respect to the coloring } c_1\textit{, there exists a } (5, 3, xa) \textit{ critical path.} \tag{4.12}$$

Recall that by $Claim$ 4.27, with respect to the coloring c_0, there existed a $(2, 3, xa)$ critical path. After recoloring the edge xb with color 5 (i.e., with respect to the coloring c_1), the $(2, 3, xa)$ critical path gets curtailed to a $(2, 3, a, b)$ maximal bichromatic path that ends at vertex b with an edge colored 3. Note that $(2, 3, a, b)$ maximal bichromatic path does not contain the vertex y, since if y is in this path, then it is an internal vertex and thus both colors $2, 3 \in F_y$, a contradiction ($2 \notin F_b$). Thus noting that $c_1(b, l_1) = 3$, we have,

With respect to the coloring c_1, there exists a $(2, 3, a, b)$ maximal bichromatic path that ends at vertex b with an edge colored 3. This path contains the edge bl_1 but does not contain vertex y. (4.13)

In view of $Claim$ 4.28, we have $S_{bl_1} = S_{bl_2} = \{2, 5, 6\}$. The 5-tuple $(b, l_1, l_2, \{l_3\}, \{x\})$ is in $configuration\ A$. Let,

$$c_2 = ColorExchange(c_1, bl_1, bl_2)$$

By $Fact$ 4.7 if there is any bichromatic cycle, recalling that $c_2(x, b) = 5$, there has to be either $(3, 5)$ or $(4, 5)$ bichromatic cycle that passes through vertex x. But any cycle that passes through vertex x should contain edge xa also. Since $c_2(x, a) = 1$, this is a contradiction and we infer that c_2 is acyclic.

Note that by (4.13) there existed $(2, 3, a, b)$ maximal bichromatic path containing the edge bl_1 with respect to the coloring c_1. Since the color of edge bl_1 is changed in c_2, this path gets curtailed to a $(2, 3, a, l_1)$ maximal bichromatic path which now ends at the vertex l_1 since $3 \notin F_{l_1}$ with respect to the coloring c_2. Note that it still does not contain vertex y. Thus we have,

With respect to the coloring c_2, there exists a $(2, 3, a, l_1)$ maximal bichromatic path which does not contain vertex y. (4.14)

But before $color\ exchange$ (i.e., with respect to the coloring c_1) by (4.12) there was a

$(5, 3, xa)$ critical path. Clearly this path passes through the vertex b. Thus by $Lemma$ 2.8, the $(5, 3, xa)$ critical path, does not exist after the color exchange (with respect to the coloring c_2) (It easy to see that l_1, $l_2 \notin \{x, a\}$ since $1 \notin F_{l_1}$, F_{l_2} but $1 \in F_x$, F_a. Therefore $Lemma$ 2.8 can be applied). Thus we have,

With respect to the coloring c_2, there does not exists any $(5, 3, xa)$ critical path. (4.15)

Now let

$$c_3 = Recolor(c_2, xa, 3)$$

By $Claim$ 4.25, $s_{xa} = \{2, 5, 6\}$ with respect to the coloring c_0 and $s_{xa} = \{2, 5, 6\}$ even with respect to the coloring c_2. Thus color 3 is candidate for edge xa since $3 \notin S_{xa}$ and $c_2(x, b) = 5$. Coloring c_3 is also acyclic since if there is any bichromatic cycle containing edge xa then it should also contain edge xb. But $c_3(x, b) = 5$ and $c_3(x, a) = 3$. Thus it has to be a $(3, 5)$ bichromatic cycle, implying that there exists a $(5, 3, xa)$ critical path with respect to the coloring c_2, a contradiction (by (4.15)).

Note that by (4.14) there existed $(2, 3, a, l_1)$ maximal bichromatic path with respect to the coloring c_2. Since the color of edge xa is changed in c_3 to color 3, it is easy to see that this path gets extended to a $(3, 2, x, l_1)$ maximal bichromatic path which now starts at the vertex x since $2 \notin F_x$ with respect to the coloring c_3. Note that it still does not contain vertex y.

Now with respect to the coloring c_3, $F = \{1, 3, 4, 5\}$ and $F_x \cap F_y = \{3\}$. Thus color 2 is a candidate for the edge xy. Since $(2, 3, x, l_1)$ maximal bichromatic path contains vertex x and does not contain vertex y, by $Fact$ 2.1 there can not be $(2, 3, xy)$ critical path. Thus by $Fact$ 4.1 color 2 is valid for the edge xy.

■

4.3 Comments

The following is obvious from $Theorem$ 4.9:

Corollary 4.29. *Let G be a graph with maximum degree $\Delta \leq 4$. Then $a'(G) \leq 7$.*

Proof: If $\Delta(G) \leq 4$, then $m \leq 2n$ for each connected component. If $m \leq 2n-1$, by Theorem 4.9 $a'(G) \leq 6$ for each connected component. Otherwise if $m = 2n$, we can remove an edge from each connected component and color the resulting graph with at most 6 colors. Now the removed edges of each component could be colored using a new color. Thus $a'(G) \leq 7$. $\square$

Remark: There exist graphs with $\Delta(G) \leq 4$ that require at least 5 colors to be acyclically edge colored. For example, any graph with $\Delta(G) = 4$ and $m = 2n - 1$ requires 5 colors. Also by using a simple counting argument we can show that $K_{2,2,2}$ (complete tripartite graph with 2 vertices in each part) needs at least 6 colors to be acyclically edge colored (see [12]). But we do not know whether there exist any graphs with $\Delta(G) \leq 4$ that needs 7 colors to be acyclically edge colored. Thus we feel that the bound of $Corollary$ 4.29 and $Theorem$ 4.9 can be improved.

Chapter 5

2-degenerate Graphs

In this chapter, we look at 2-$degenerate$ graphs.

Definition 5.1. *A graph G is called k-$degenerate$ if any induced subgraph of G, has a vertex of degree at most k. For example, planar graphs are 5-degenerate, forests are 1-degenerate.*

5.1 Previous Results

The earliest result on acyclic edge coloring of 2-degenerate graphs was by Card and Roditty [17], where they proved that $a'(G) \leq \Delta + k - 1$, where k is the maximum edge connectivity, defined as $k = \max_{u,v \in V(G)} \lambda(u,v)$, where $\lambda(u,v)$ is the edge- connectivity of the pair u,v. Note that here k can be as high as Δ. Muthu,Narayanan and Subramanian [34] proved that $a'(G) \leq \Delta + 1$ for outerplanar graphs which are a subclass of 2-degenerate graphs and posed the problem of proving the conjecture for 2-degenerate graphs as an open problem. In fact they have informed us that very recently they have also derived an upper bound of $\Delta + 1$ for series-parallel graphs [35], which is a slightly bigger subclass of 2-degenerate graphs. Connected non-regular subcubic graphs are 2-degenerate graphs with $\Delta = 3$. Recently Basavaraju and Chandran [10] proved that connected non-regular subcubic graph can be acyclically edge colored using $\Delta + 1 = 4$ colors. Another two interesting subclasses of 2-degenerate graphs are *planar graphs of girth 6* and *circle graphs of girth 5* [1]. As far as we know, nothing much is known about the acyclic edge chromatic number of these graphs.

5.2 The Theorem

Theorem 5.2. *Let G be a 2-degenerate graph with maximum degree Δ, then $a'(G) \leq \Delta + 1$.*

Proof: We prove the theorem by way of contradiction. Let G be a 2-degenerate graph with n vertices and m edges which is a minimum counter example for the theorem statement. Then the theorem is true for all 2-degenerate graphs with at most $m-1$ edges. To prove the theorem for G, we may assume that G is connected. We may also assume that the minimum degree, $\delta(G) \geq 2$, since otherwise if there is a vertex v, with $degree(v) = 1$, we can easily extend the acyclic edge coloring of $G-e$ (where e is the edge incident on v) to G. Keeping the assumption that G is a minimum counter example in mind we will show that any partial coloring c of G should satisfy certain properties which in turn will lead to a contradiction.

Selection of the Primary Pivot: Let $W_0 = \{z \in V(G) \mid degree_G(z) = 2\}$. Since G is 2-degenerate $W_0 \neq \emptyset$. We may assume that $V - W_0 \neq \emptyset$ because otherwise, G is a cycle and it is easy to see that it is $\Delta + 1 = 3$ acyclically edge colorable. Thus $\Delta(G) \geq 3$. Let $G' = G[V - W_0]$ and $W_1 = \{z \in V(G') \mid degree_{G'}(z) \leq 2\}$. By the definition of 2-degeneracy there exists at least one vertex of degree at most 2 in G' and thus $W_1 \neq \emptyset$.

Let $V' = V(G')$. If $V' - W_1 \neq \emptyset$, then there exists at least one vertex of degree at most 2 in $G'[V' - W_1]$. Let $G'' = G[V' - W_1]$ and $W_2 = \{z \in V(G'') \mid degree_{G''}(z) \leq 2\}$. Let $q \in W_2$. Clearly $N_G(q) \cap W_1 \neq \emptyset$ and let $x \in N_G(q) \cap W_1$. On the other hand if $V' - W_1 = \emptyset$, then let $x \in W_1$. We call x the $Primary\ Pivot$, since x plays an important role in our proof. Let $N'_G(x) = N_G(x) \cap W_0$ and $N''_G(x) = N_G(x) - N'_G(x)$ = $N_{G'}(x)$. Since $x \in W_1$, it is easy to see that $|N''_G(x)| \leq 2$ and $|N'_G(x)| \geq 1$.

Let $N'_G(x) = \{y_1,\ y_2, \ldots, y_t\}$. Also $\forall y_i$, let $N_G(y_i) = \{x, y'_i\}$ (See $figure$ 5.1). $\forall y_i$, let G_i denote the graph obtained by removing the edge (x, y_i) from the graph G. Let $N'_{G_i}(x) = N'_G(x) - \{y_i\}$ and $N''_{G_i}(x) = N_{G_i}(x) - N'_{G_i}(x)$. By the minimum choice of G, graph G_i is $\Delta + 1$ acyclically edge colorable. Let c_i be a valid coloring of G_i and thus a partial coloring of G. We denote the set of colors by $C = \{1, 2, \ldots, \Delta + 1\}$.

Comment: Note that the figures given in this paper are only for providing visual aid for the reader. They do not capture all possible configurations.

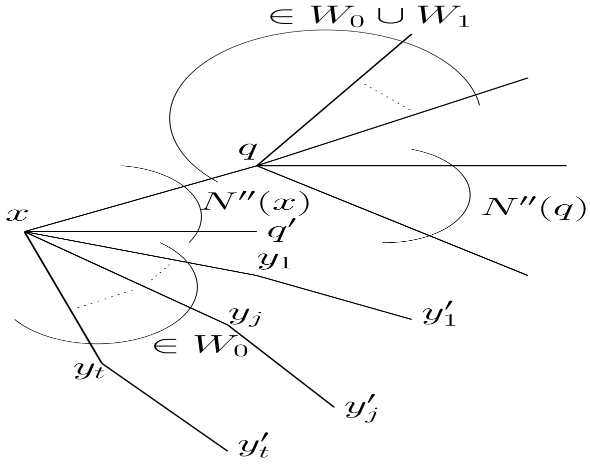

Figure 5.1: Vertex x and its neighbors

5.2.1 Properties of any valid coloring c_i of G_i

Let $F_x(c_i) = \{c_i(x, z)|z \in N_{G_i}(x)\}$. Let $F'_x(c_i) = \{c_i(x, z)|z \in N'_{G_i}(x)\}$ and $F''_x(c_i) = \{c_i(x, z)|z \in N''_{G_i}(x)\}$. Note that $F_x(c_i)$ is the disjoint union of $F'_x(c_i)$ and $F''_x(c_i)$ and also $|F''_x(c_i)| \leq 2$.

Lemma 5.3. *With respect to any valid coloring c_i of G_i, $c_i(y_i, y'_i) \in F''_x(c_i)$.*

Proof: It is easy to see that $c_i(y_i, y'_i) \in F_x(c_i)$. Otherwise all the candidate colors are valid for the edge xy_i, since any cycle involving the edge xy_i will contain the edge $y_iy'_i$ as well as an edge incident on x in G_i and thus the cycle will have at least 3 colors. Suppose $c_i(y_i, y'_i) \in F'_x(c_i)$. Clearly we have $|F_x(c_i) \cup \{c_i(y_i, y'_i)\}| \leq \Delta - 1$. Thus there are at least two *candidate colors* for the edge xy_i. Let $y_j \in N'_{G_i}(x)$ be the vertex such that $c_i(y_i, y'_i) = c_i(x, y_j)$. When we color edge xy_i there is a possibility of a bichromatic cycle only if we assign $c_i(y_j, y'_j)$ to the edge xy_i since $degree_{G_i}(y_j) = 2$. But since we have at least two *candidate colors* for edge xy_i, this situation can easily be avoided. We infer that $c_i(y_i, y'_i) \in F''_x(c_i)$. ■

Lemma 5.4. *With respect to any valid coloring c_i of G_i, $|F''_x(c_i)| = 2$*

Proof: Suppose not. Then $|F''_x(c_i)| \leq 1$. Since $|F_{y'_i}(c_i)| \leq \Delta$, we have at least one *candidate color* for the edge $y_iy'_i$. Note that any *candidate color*, is $valid$ for the edge $y_iy'_i$ in G_i since y_i is a pendant vertex in G_i. Let c'_i be the valid coloring obtained by recoloring the edge $y_iy'_i$ with a candidate color. By $Lemma$ 5.3, we have $c_i(y_i, y_i') \in F_x''(c_i)$. Clearly since $|F_x''(c_i)| \leq 1$ and $c_i'(y_i, y_i') \neq c_i(y_i, y_i')$, we can infer that $c_i'(y_i, y_i') \notin F_x''(c_i) = F_x''(c_i')$, a contradiction to $Lemma$ 5.3. ■

An immediate consequence of $Lemma$ 5.4 is that $|N_G''(x)| = 2$. Moreover by the way we have selected vertex x at least one of them should belong to $W_1 \cup W_2$. We make the following assumption:

Assumption 5.5. *With respect to any valid coloring c_i of G_i, without loss of generality let $F_x''(c_i) = \{1, 2\}$ and $N_G''(x) = \{q, q'\}$. Thus $\{c_i(x, q), c_i(x, q')\} = \{1, 2\}$. Also without loss of generality we assume that $q \in W_2 \cup W_1$ (see $figure$ 5.1).*

Lemma 5.6. *With respect to any valid coloring c_i of G_i, colors $1,\ 2 \notin S_{y_i y_i'}$.*

Proof: Since $|F_{y_i'}(c_i)| \leq \Delta$, we have at least one *candidate color* $\gamma \neq c_i(y_i, y_i')$ for the edge $y_i y_i'$. Note that γ is $valid$ for the edge $y_i y_i'$ in G_i since y_i is a pendant vertex in G_i. Let c_i' be the valid coloring obtained by recoloring the edge $y_i y_i'$ with γ. Now since c_i as well as c_i' are valid, by $Lemma$ 5.3, we have $\{c_i(y_i, y_i'), c_i'(y_i, y_i')\} = F_x''(c_i) = \{1, 2\}$ (by $Assumption$ 5.5). Since $c_i(y_i, y_i') \notin S_{y_i y_i'}$ and $c_i'(y_i, y_i') \notin S_{y_i y_i'}$, we have $1,\ 2 \notin S_{y_i y_i'}$. ■

Let $C' = C - \{1, 2\}$. For each color $\gamma \in C'$, we define a graph $G_{i,\gamma}$ as below:

$$G_{i,\gamma} = \begin{cases} G_i & \text{if } \gamma \in C' - F_x'(c_i) \\ G_i - xy_a,\ where\ c_i(x, y_a) = \gamma & \text{if } \gamma \in F_x'(c_i) \end{cases}$$

Also let $c_{i,\gamma}$ be the valid coloring of $G_{i,\gamma}$ derived from c_i of G_i, that is by discarding the color of the edge xy_a, where y_a is the vertex such that $c_i(x, y_a) = \gamma$. Also if $c_{i,\gamma}$ is a valid coloring of $G_{i,\gamma}$, then $c_{i,\gamma}$ is said to be derivable from c_1 if we can extend the coloring $c_{i,\gamma}$ of $G_{i,\gamma}$ to the coloring c_1 of G_1. Also note that even though we define these graphs $G_{i,\gamma}$, we always have the original graph in mind when using definitions like critical paths, which are defined with respect to an edge in the graph.

Lemma 5.7. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall \gamma \in C' - F_x'(c_i)$, $\exists (\mu, \gamma, xy_i)$ critical path, where $\mu = c_i(y_i, y_i')$.*

Proof: Recall that when $\gamma \in C' - F_x'(c_i)$, we have $G_{i,\gamma} = G_i$ and hence $c_{i,\gamma} = c_i$. Suppose if there is no (μ, γ, xy_i) critical path, where $\gamma \in C' - F'_x(c_i)$, then by $Fact$ 2.5 color γ is valid for the edge xy_i. Thus we get a valid coloring of G, a contradiction. ■

Lemma 5.8. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall \gamma \in C' - F'_x(c_i)$, $\exists(\nu, \gamma, x, y'_i)$ maximal bichromatic path, where $\{\nu\} = \{1, 2\} - \{c_i(y_i, y'_i)\}$.*

Proof: Recall that when $\gamma \in C' - F'_x(c_i)$, we have $G_{i,\gamma} = G_i$ and hence $c_{i,\gamma} = c_i$. Suppose there is no (ν, γ, x, y'_i) maximal bichromatic path, where $\gamma \in C' - F'_x(c_i)$. By $Lemma$ 5.6, color ν is a candidate for the edge $y_iy'_i$. Now recolor the edge $y_iy'_i$ with color ν to get a valid coloring c'_i of G_i. Since by our assumption that there is no (ν, γ, x, y'_i) maximal bichromatic path with respect to $c_{i,\gamma} = c_i$, there cannot be any (ν, γ, xy_i) critical path with respect to the coloring c'_i, a contradiction to $Lemma$ 5.7 (Note that the color μ discussed in Lemma 5.7 and assumption is same as $\nu = c'_i(y_i, y'_i)$ in c'_i). ■

Assumption 5.9. *Since $|F_x(c_i)| \leq \Delta - 1$, we have $|C - F_x(c_i)| \geq 2$. Since $C - F_x(c_i) = C' - F'_x(c_i)$, we have $|C' - F'_x(c_i)| \geq 2$. Thus $degree_{G_i}(y'_i) \geq 3$ and hence $degree_G(y'_i) \geq 3$. Let $\alpha, \beta \in C' - F'_x(c_i)$.*

Lemma 5.10. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall \gamma \in F'_x(c_i)$, $\exists(\mu, \gamma, xy_i)$ critical path, where $\mu = c_i(y_i, y'_i)$.*

Proof: Let $c_i(x, y_j) = \gamma$, where $\gamma \in F'_x(c_i)$. Suppose if there is no (μ, γ, xy_i) critical path, then by $Fact$ 2.5 color γ is valid for the edge xy_i with respect to the coloring $c_{i,\gamma}$. Color the edge xy_i with color γ to get a valid coloring d of $G - \{xy_j\}$.

Now we will show that we can extend the coloring d of $G - \{xy_j\}$ to a valid coloring of the graph G by giving a valid color for the edge xy_j, leading to a contradiction of our assumption that G was a minimum counter example. We claim the following:

Claim 5.11. *With respect to the coloring d, either color α or β is valid for the edge xy_j (Recall that $\alpha, \beta \in C' - F'_x(c_i)$ by $Assumption 5.9$)*

Proof: Without loss of generality, let $d(y_j, y'_j) = \eta$. Note that $\eta \neq \gamma = c_i(x, y_j)$. Now if,

1. $\eta \notin F_x(c_i)$. In view of Assumption 5.9, α, $\beta \notin F_x(c_i)$. Noting that η cannot be equal to both α and β, without loss of generality, let $\eta \neq \alpha$. Then color the edge (x, y_j) with color α to get a proper coloring d'. If a bichromatic cycle gets formed, then it should

contain the edge xy_j and also involve both the colors η and α since $degree_G(y_j) = 2$. But since $\eta \notin F_x(c_i)$, such a bichromatic cycle is not possible. Thus the coloring d' is valid.

2. $\eta \in \{1, 2\} = \{\mu, \nu\} = F''_x(c_i)$. Recolor the edge xy_j with color α to get a coloring d'. We claim that the coloring d' is valid. This is because if it is not valid, then there has to be a (α, η) bichromatic cycle containing the edge xy_j with respect to d'. This implies that there has to be a (η, α, xy_j) critical path with respect to the coloring d and hence with respect to the coloring $c_{i,\gamma}$ (Note that the coloring d is obtained from $c_{i,\gamma}$ just by giving the color γ to the edge xy_i and $\eta, \gamma \neq \alpha, \beta$).

   If $\eta = \mu$, this means that there was a $(\eta = \mu, \alpha, xy_j)$ critical path with respect to $c_{i,\gamma}$. But this is not possible by $Fact$ 2.1 since there is already a (μ, α, xy_i) critical path with respect to $c_{i,\gamma}$ (by $Lemma$ 5.7) and $y_i \neq y_j$.

   Thus $\eta = \nu$. This means that there has to be a $(\eta = \nu, \alpha, xy_j)$ critical path with respect to $c_{i,\gamma}$. But this is not possible by $Fact$ 2.1 since there is already a (ν, α, x, y'_i) maximal bichromatic path with respect to $c_{i,\gamma}$ (by $Lemma$ 5.8) and $y'_i \neq y_j$ ($y'_i \neq y_j$ since by $Assumption$ 5.9, $degree_{G_i}(y'_i) \geq 3$. But $degree_{G_i}(y_j) = 2$). Thus there cannot be any bichromatic cycles with respect to the coloring d'. Thus the coloring d' is valid.

3. $\eta \in F'_x(c_i)$. Let $y_k \in N'_G(x)$ be such that $d(x, y_k) = \eta$. With respect to colors $\{\alpha, \beta\}$, without loss of generality let $d(y_k, y'_k) \neq \beta$. Recall that $d(y_j, y'_j) = \eta$. Now recolor the edge xy_j with color β to get a coloring d'. Now if a bichromatic cycle gets formed, then it should contain the edge xy_j and also involve both the colors η and β. Thus the bichromatic cycle should contain the edge xy_k. Since $degree_G(y_k) = 2$, the bichromatic cycle should contain the edge $y_k y'_k$. But by our assumption, $c_i(y_k, y'_k) \neq \beta$, a contradiction. Thus the coloring d' is valid.

Hence either color α or β is valid for the edge xy_j.

□

Thus we have a valid coloring (i.e, d') for the graph G, a contradiction. ■

Lemma 5.12. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall\gamma \in F'_x(c_i)$, $\exists(\nu, \gamma, x, y'_i)$ maximal bichromatic path, where $\{\nu\} = \{1, 2\} - \{c_i(y_i, y'_i)\}$.*

Proof: Suppose if there is no (ν, γ, x, y'_i) maximal bichromatic path, where $\gamma \in F'_x(c_i)$, then by $Lemma$ 5.6, color ν is a candidate for the edge $y_iy'_i$. Now recolor the edge $y_iy'_i$ with color ν to get a valid coloring $c'_{i,\gamma}$ of G_i. Since by our assumption that there is no (ν, γ, x, y'_i) maximal bichromatic path with respect to $c_{i,\gamma}$, there cannot be any (ν, γ, xy_i) critical path with respect to the coloring $c'_{i,\gamma}$, a contradiction to $Lemma$ 5.10 (Note that the color μ discussed in Lemma 5.10 and assumption is same as $\nu = c'_{i,\gamma}(y_i, y'_i)$ in $c'_{i,\gamma}$).

■

Critical Path Property: In the rest of the paper we will have to repeatedly use the properties (namely the presence of (μ, γ, xy_i) critical path in $G_{i,\gamma}$, where $\mu = c_{i,\gamma}(y_i, y'_i)$) described by $Lemma$ 5.7 and $Lemma$ 5.10. Therefore we will name these properties as the *Critical Path Property* of the graph $G_{i,\gamma}$.

If c_i is any valid coloring of G_i, then in $G_{i,\gamma}$, $\forall \gamma \in C'$, by $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10) there exists a (μ, γ, xy_i) critical path and by $Lemma$ 5.8 and $Lemma$ 5.12 there exists a (ν, γ, x, y'_i) maximal bichromatic path, where $\mu = c_i(y_i, y'_i)$ and $\{\nu\} = F''_x(c_i) - \{\mu\}$. Recall that $|S_{ab}| \leq \Delta - 1$ for any $ab \in E$. As an immediate consequence we have,

$$S_{xq} = S_{xq'} = S_{y_iy'_i} = C - \{1, 2\} = C'. \tag{5.1}$$

In view of (5.1), we have

$$|S_{xq}| = |S_{xq'}| = |S_{y_iy'_i}| = |C'| = \Delta - 1. \tag{5.2}$$

Lemma 5.13. *Let c_i be any valid coloring of G_i. Let $\mu = c_i(y_i, y'_i) \in \{1, 2\}$. Also let $y_j \in N'_G(x) - \{y_i\}$. Then $\forall \gamma \in C'$, the (μ, γ, xy_i) critical path in $G_{i,\gamma}$ does not contain the vertex y_j.*

Proof: Suppose there exists a (μ, γ, xy_i) critical path that contains the vertex y_j, then y_j cannot be an end vertex as $y_i \neq y_j$. Thus y_j is an internal vertex. Now since $degree_G(y_j) = 2$, the (μ, γ, xy_i) critical path should contain the edge xy_j as well. But the (μ, γ, xy_i) critical path ends at vertex x with color μ which implies $c_i(x, y_j) = \mu$, a contradiction since

$c_i(x, y_j) \notin \{1, 2\} = \{\mu, \nu\}$. ■

Lemma 5.14. *Let c_i be any valid coloring of G_i and let $u \in \{q, q'\}$. Let $\mu = c_i(y_i, y_i') = c_i(x, u) \in \{1, 2\}$ and $\nu = \{1, 2\} - \{\mu\}$. Then $\forall\gamma \in C'$, the (μ, γ, xy_i) critical path in $G_{i,\gamma}$ has length at least five.*

Proof: Suppose not. Then the (μ, γ, xy_i) critical path has length three which implies that the vertices in the critical path are x, u, y_i', y_i in that order. Thus $\mu \in F_u(c_i)$ and $F_u(c_i) = S_{xu} \cup \{\mu\}$. Now change the color of the edge $y_i'y_i$ to ν. It is proper since by 5.1, we have $\{1, 2\} = \{\mu, \nu\} \notin S_{y_iy_i'}$. It is valid since y_i is a pendant vertex in $G_{i,\gamma}$. Now in view of $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10) there has to be a (ν, γ, xy_i) critical path that passes through the vertex y_i' with respect to this new coloring. Since $c_{i,\gamma}(u, y_i') = \gamma$, this (ν, γ, xy_i) critical path should contain vertex u as an internal vertex, which implies that color $\nu \in F_u(c_i)$. Recalling that $F_u(c_i) = S_{xu} \cup \{\mu\}$, we have $\nu \in S_{xu}$, a contradiction in view of (5.1). Thus the (μ, γ, xy_i) critical path has length at least five with respect to the coloring $c_{i,\gamma}$ of $G_{i,\gamma}$. ■

5.2.2 The structure of the minimum counter example in the vicinity of the primary pivot, x

Lemma 5.15. *The minimum counter example G satisfies the following properties,*

(a) $\forall u, v \in N_G(x)$, $(u, v) \notin E(G)$.

(b) $\forall y_i \in N_G'(x)$ *and* $\forall v \in N_G(x) - \{y_i\}$, *we have* $(v, y_i') \notin E(G)$.

Proof: To prove (a) we consider the following cases:
case 1.1: $u, v \in N_G'(x)$
Let $u = y_k$ and $v = y_j$. Now if $u \in N_G(v)$, then $u = y_j'$. Recalling that $\Delta(G) \geq 3$, in view of (5.2), we have $degree_G(u) = degree_G(y_j') \geq 3$. But $degree_G(u) = degree_G(y_k) = 2$, a contradiction.

case 1.2: $u, v \in N_G''(x)$
Then we need to show that $q' \notin N_G(q)$. To see this consider the coloring c_i of graph G_i. We

know that $\{c_i(x,q), c_i(x,q')\} = \{\mu, \nu\}$. Without loss of generality let $c_i(x,q) = c_i(y_i, y_i') = \mu$. Note that by (5.2), we have $S_{xq} = C'$. If $q' \in N_G(q)$, then $c_i(q,q') \in C'$. Let $c_i(q,q') = \gamma \notin \{\mu, \nu\}$. Now in $G_{i,\gamma}$, the (μ, γ) maximal bichromatic path that starts at vertex x contains only edges xq and qq' since $\mu \notin F_{q'}(c_i)$ (by (5.2)). Thus by $Fact$ 2.1, there cannot be a (μ, γ, xy_i) critical path in $G_{i,\gamma}$, a contradiction to $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10). Thus $q' \notin N_G(q)$.

case 1.3: $u \in N''_G(x)$ ***and*** $v \in N'_G(x)$
Let $v = y_i$. Then we have to show that $y_i' \notin N''_G(x) = \{q, q'\}$. To see this consider the coloring c_i of graph G_i. Recall that $\{c_i(x,q), c_i(x,q')\} = \{\mu, \nu\}$. Without loss of generality let $c_i(x,q) = c_i(y_i, y_i') = \mu$. Now if $y_i' = q$, then we have $c(q, y_i) = c_i(y_i', y_i) = \mu$, a contradiction since $c(x,q) = \mu$. On the other hand if $y_i' = q'$, then $c(q', y_i) = c_i(y_i', y_i) = \mu$. This means that $\mu \in S_{xq'}$, a contradiction in view of (5.1). Thus $y_i' \neq q, q'$.

Thus $\forall u, v \in N_G(x)$, we have $(u,v) \notin E(G)$

To prove (b) we consider the following cases:
case 2.1: $v \in N'_{G_i}(x)$
Let $v = y_j \in N'_G(x)$. If $(v, y_i') = (y_j, y_i') \in E(G)$, then $y_i' = y_j'$. Consider the coloring c_j of graph G_j. Let $c_j(y_j, y_j') = \mu$. Recall that by (5.2), we have $S_{y_j y_j'} = C'$. If $y_i' = y_j'$, then $c_j(y_j', y_i) \in C'$. Let $c_j(y_j', y_i) = \gamma$. Now in $G_{j,\gamma}$, the (μ, γ) maximal bichromatic path that starts at vertex y_j contains only edges $y_j y_j'$, $y_j' y_i$ and thus ends at vertex y_i since $\mu \notin F_{y_i}(c_j)$. This is because $N_{G_{j,\gamma}}(y_i) = \{y_j', x\}$ and we have $c_j(y_j', y_i) = \gamma$ and $c_j(x, y_i) \neq \mu$(since by $Assumption$ 5.5, $\mu \in c_j(x,q), c_j(x,q')\}$). Thus by $Fact$ 2.1, there cannot be a (μ, γ, xy_j) critical path in $G_{j,\gamma}$, a contradiction to $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10). Thus $y_j' \neq y_i'$.

case 2.2: $v \in N''_G(x) = \{q, q'\}$
Then we have to show that $y_i' \notin N_G(q) \cup N_G(q')$. To see this consider the coloring c_i of graph G_i. Recall that $\{c_i(x,q), c_i(x,q')\} = \{\mu, \nu\}$. Without loss of generality let $c_i(x,q) = c_i(y_i, y_i') = \mu$. Suppose $y_i' \in N_G(q)$, then we have $c(y_i', q) \in S_{xq}$. Thus by (5.1), we have $c_i(y_i', q) \neq \nu$. Now there exists a $(\mu, c_i(y_i', q) \neq \nu, xy_i)$ critical path of length 3, a contradiction to $Lemma$ 5.14. Now if $y_i' \in N_G(q')$, then we recolor the edge $y_i y_i'$ with color ν to get a valid coloring c_i'. Now there exists a $(\nu, c_i(y_i', q'), xy_i)$ critical path of length 3, a contradiction to

$Lemma$ 5.14. Thus $y'_i \notin N_G(q) \cup N_G(q')$.

Thus $\forall y_i \in N'_G(x)$ and $\forall v \in N_G(x) - \{y_i\}$, we have $(v, y'_i) \notin E(G)$.

■

5.2.3 Modification of valid coloring c_1 of G_1 to get valid coloring c_j of G_j

Assumption 5.16. *Let c_1 be a valid coloring of G_1 and without loss of generality let $c_1(x, q) = 1$, $c_1(x, q') = 2$ and $c_1(y_1, y'_1) = \mu = 1$.*

Remark: In view of $Assumption$ 5.16, the $Critical\ Path\ Property$ with respect to the coloring c_1 of G_1 reads as follows: With respect to the coloring $c_{1,\gamma}$, there exists a $(1, \gamma, xy_1)$ critical path, for all $\gamma \in C'$.

Let f_1 be the coloring of G_1 obtained from c_1 by exchanging the colors of the edges xq and xq'. Also for $\gamma \in C'$, we define the coloring $f_{1,\gamma}$ as the coloring obtained from $c_{1,\gamma}$ by exchanging the colors with respect to the edges xq and xq'. Note that $f_{1,\gamma}$ can be obtained from f_1 just by discarding the γ colored edge incident on vertex x for $\gamma \in F'_x(f_1)$.

Claim 5.17. *The coloring f_1 is proper but is not valid.*

Proof: The coloring f_1 is proper since in view of (5.1), $2 \notin S_{xq}$ and $1 \notin S_{xq'}$. Suppose the coloring f_1 is valid. Let γ be a candidate color for the edge xy_1. Clearly $\gamma \in C - F_x(f_1)$. Now since f_1 is proper, taking $u = x$, $i = q$, $j = q'$, $ab = xy_1$, $\lambda = 1$ and $\xi = \gamma$, $Lemma$ 2.8 can be applied. There existed a $(1, \gamma, xy_1)$ critical path with respect to coloring c_1. By $Lemma$ 2.8, we infer that there cannot be any $(1, \gamma, xy_1)$ critical path with respect to the coloring f_1. Thus by Fact 2.5, candidate color γ is valid for the edge xy_1. Thus we have obtained a valid coloring for the minimum counter example G, a contradiction. □

By $Claim$ 5.17, there exist bichromatic cycles with respect to the coloring f_1. It is clear that each bichromatic cycle with respect to f_1 has to contain either the edge xq or xq' since we have changed only the colors of the edges xq and xq' to get the coloring f_1 from c_1. Thus each such bichromatic cycle should be either a $(1, \gamma)$ bichromatic cycle or a $(2, \gamma)$ bichromatic cycle. Note that each of these bichromatic cycles should pass through the vertex x. Moreover observe that there cannot be any $(1, 2)$ bichromatic cycle since color $1 \notin S_{xq}$ with respect to f_1 in view of (5.1). Thus $\gamma \in F'_x(f_1)$. From this we infer that $|F'_x(f_1)| \geq 1$. Recalling $Assumption$ 5.9, we have $|C - F_x(f_1)| \geq 2$. It follows that $|C'| \geq 3$. Thus we have,

$$\Delta(G) \geq degree_{G_1}(q) \geq |S_{xq}| + 1 \geq |C'| + 1 \geq 4. \tag{5.3}$$

Let

$C_1 = C_1(f_1) = \{\gamma \in F'_x(c_1) |\ \exists(1, \gamma)$ bichromatic cycle with respect to coloring $f_1\}$.

$C_2 = C_2(f_1) = \{\gamma \in F'_x(c_1) |\ \exists(2, \gamma)$ bichromatic cycle with respect to coloring $f_1\}$.

Note that from the discussion above, any bichromatic cycle with respect to the coloring f_1 contains a vertex $y_i \in N'_{G_1}(x)$. But $degree_{G_1}(y_i) = 2$ and therefore $|S_{xy_i}| = 1$. Thus S_{xy_i} contains exactly one of the color 1 or 2. Thus with a fixed color $\gamma \in C_1 \cup C_2$ there exists exactly one of $(1, \gamma)$ or $(2, \gamma)$ bichromatic cycle, which implies that the sets C_1 and C_2 cannot have any element in common (See $figure$ 5.2). Thus we have,

$$C_1 \cap C_2 = \emptyset. \tag{5.4}$$

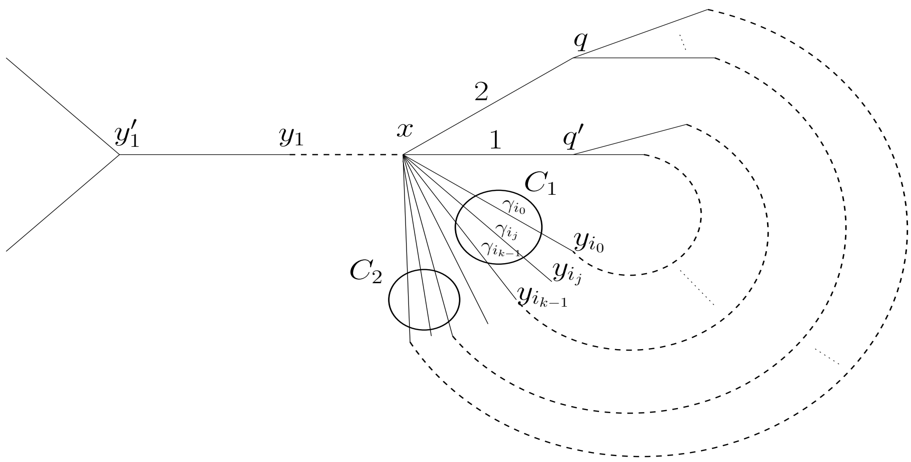

Figure 5.2: Bichromatic cycles of C_1 and C_2

Recall that in view of $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10), for a coloring $c_{1,\gamma}$ of $G_{1,\gamma}$, $\forall\gamma \in C'$, there exists a $(1, \gamma, xy_1)$ critical path. With respect to the new coloring $f_{1,\gamma}$, since the colors of only edges xq and xq' are changed, this path starts from y_1 and reaches the vertex q. But since color 1 is not present at vertex q with respect to the coloring $f_{1,\gamma}$, the bichromatic path ends at vertex q. Thus the $(1, \gamma, xy_1)$ critical path with respect to coloring $c_{1,\gamma}$ gets curtailed to a $(\gamma, 1, q, y_1)$ maximal bichromatic path with respect to $f_{1,\gamma}$. Also note that in view of $Lemma$ 5.14 the length of this $(\gamma, 1, q, y_1)$ maximal bichromatic path is at least four. This is true for the coloring f_1 also i.e., there exists a $(\gamma, 1, q, y_1)$ maximal bichromatic path with respect to f_1. To see this observe that f_1 is obtained from $f_{1,\gamma}$ by putting back the edge xy_a, where $c_1(x, y_a) = \gamma$. This cannot alter the $(\gamma, 1, q, y_1)$ maximal bichromatic path since x does not belong to this path and also $y_a \neq y_1$. Also in view of Lemma 5.13, none of the above maximal bichromatic paths contain vertex y_j, $\forall y_j \in N'_G(x) - \{y_1\}$. Thus the coloring f_1 satisfies the following property which we name as $Property\ A$:

Property A: A partial coloring of G is said to satisfy $Property\ A$ iff $\forall\gamma \in C - \{1, 2\}$, there exists a $(\gamma, 1, q, y_1)$ maximal bichromatic path of length at least four. Moreover none of the above maximal bichromatic paths contain vertex x or vertex y_i, where $y_i \in N'_G(x) - \{y_1\}$.

Claim 5.18. *There exists a proper coloring f'_1 obtained from f_1 such that $\forall i \in \{1, 2\}$, $|C_i| \leq 1$, where $C_i = C_i(f'_1)$. Moreover f'_1 satisfies $Property\ A$.*

Proof: If $|C_1| \leq 1$ and $|C_2| \leq 1$, then let $f'_1 = f_1$. If $|C_1| \leq 1$, then let $f''_1 = f_1$. Otherwise if $|C_1| \geq 2$, then let $C_1 = \{\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_{k-1}}\}$ and also let y_{i_j} be the vertex such that $f_1(x, y_{i_j}) = \gamma_{i_j}$, $\forall j \in \{0, 1, 2, \ldots k-1\}$ (see $Figure$ 5.2). Now let the coloring f''_1 be defined as $f''_1(x, y_{i_j}) = \gamma_{i_l}$, where $l = j + 1 (mod\ k)$, $\forall j \in \{0, 1, 2, \ldots k-1\}$ and $f''_1(e) = f_1(e)$ for all other edges. (*Note that we have only shifted the colors of the edges $xy_{i_0}, xy_{i_1}, \ldots, xy_{i_{k-1}}$ circularly. We call this procedure* **deranging of colors.**)

Note that we are changing only the colors of the edges xy_{i_j} for $j = 0, 1, 2, \ldots, k-1$. Also we are using only the colors $\gamma_{i_j} \in C_1$ for recoloring. Since with respect to the coloring f_1, a $(1, \gamma_{i_j})$ bichromatic cycle passed through y_{i_j} and $degree_{G_1}(y_{i_j}) = 2$, we have $S_{xy_{i_j}} = \{1\}$. Thus the coloring f''_1 is proper.

Since for all y_{i_j}, $0 \leq j \leq k-1$ we have $S_{xy_{i_j}} = \{1\}$ with respect to the coloring f''_1, it is clear that any new bichromatic cycle created (in the process of getting f''_1 from f'_1) has to be a $(1, \gamma)$ bichromatic cycle, where $\gamma \in C_1$.

We claim that the coloring f''_1 does not have any $(1, \gamma)$ bichromatic cycle for $\gamma \in C_1$. To see this consider a $\gamma \in C_1$, say γ_{i_1}. There existed a $(1, \gamma_{i_1})$ bichromatic cycle with respect to f_1. It contained the edge xy_{i_1}. Now with respect to f_1'' edge xy_{i_1} is colored with color γ_{i_2}. Thus the $(1, \gamma_{i_1})$ maximal bichromatic path which contains the vertex x has one end at vertex y_{i_1} since color γ_{i_1} is not present at the vertex y_{i_1} with respect to f_1''. Thus $(1, \gamma_{i_1})$ bichromatic cycle cannot exist with respect to the coloring f_1''. This argument works for all $\gamma \in C_1$ and thus for any color $\gamma \in C_1$, there is no $(1, \gamma)$ bichromatic cycle with respect to f_1''.

If $|C_2| \leq 1$, then $f_1' = f_1''$. Otherwise if $|C_2| \geq 2$, by performing similar recoloring (now starting with f_1'') as we did to get rid of the $(1, \gamma)$ bichromatic cycles, we can get a coloring f_1''' without any $(2, \gamma)$ bichromatic cycle. Now let $f_1' = f_1'''$. Thus we get a coloring f_1' from f_1 which has $|C_1| \leq 1$ and $|C_2| \leq 1$.

Note that we are changing only the colors of the edges xy_i, for $y_i \in N'_{G_1}(x)$. But the coloring f_1 satisfied $Property\ A$ and hence none of the $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall \gamma \in C - \{1, 2\}$, contained the vertex y_i or x. Thus these bichromatic paths have not been altered (i.e., neither $broken$ nor $extended$) by the recoloring to get f_1' from f_1. Thus the coloring f_1' satisfies $Property\ A$. □

Observation 5.19. *Note that the color of the edge* y_1y_1' *is unaltered in* f_1', *i.e.,* $f_1'(y_1, y_1') = f_1(y_1, y_1') = c_1(y_1, y_1') = 1$. *Also only the colors of certain edges incident on the vertex* y_i, *where* $y_i \in N'_G(x) - \{y_1\}$ *are modified when we obtained* f_1' *starting from* c_1. *(*This information is required later in the proof*)*.

It is easy to see that f_1' is proper but not valid. It is not valid because, if it is valid then since f_1' satisfies $Property\ A$, there are $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall \gamma \in C - \{1, 2\}$. Thus by $Fact$ 2.1, for any $\theta \in C - F_x(f_1')$, there cannot be a $(1, \theta, xy_1)$ critical path. Thus by $Fact$ 2.5, color θ is valid for the edge xy_1. Thus we have a valid coloring for the graph G, a contradiction. Thus f_1' is not valid. It implies that at least one of C_1 or C_2 is nonempty. In the next lemma we further refine the proper coloring f_1'.

Lemma 5.20. *There exists a proper coloring* h_1 *of* G_1 *obtained from* f_1' *such that there is at most one bichromatic cycle. Moreover* h_1 *satisfies* $Property\ A$.

Proof: By $Claim$ 5.18, we have $|C_1| \leq 1$ and $|C_2| \leq 1$. If exactly one of C_1, C_2 is singleton, then let $h_1 = f_1'$. Otherwise we have $|C_1| = 1$ and $|C_2| = 1$.

Assumption 5.21. *Without loss of generality let* $C_1 = \{\gamma\}$ *and* $C_2 = \{\theta\}$. *Let* $f_1'(x, y_j) = \gamma$ *and* $f_1'(x, y_k) = \theta$. *Thus* $f_1'(y_j, y_j') = 1$ *and* $f_1'(y_k, y_k') = 2$, *since there are* $(1, \gamma)$ *and* $(2, \theta)$

bichromatic cycles passing through the vertex x.

Claim 5.22. *Color* $2 \notin S_{y_j y'_j}$

Proof: Suppose not, then $2 \in S_{y_j y'_j}$. Since there is a $(1, \gamma)$ bichromatic cycle passing through y'_j, the colors 1 and γ are present at y'_j. It follows that there exists $\eta \in C - \{1, 2, \gamma\}$ missing at y'_j. Now recolor edge $y_j y'_j$ with color η to get a coloring f''_1. If the color η is valid for the edge $y_j y'_j$, then let $h_1 = f''_1$ and we are done as the situation reduces to having only one bichromatic cycle (i.e.,$|C_2| = 1$ and $|C_1| = 0$). If the color η is not valid for the edge $y_j y'_j$, then there has to be a (γ, η) bichromatic cycle that passes through vertex x. Let $f''_1(x, y_l) = \eta$. Since $degree_G(y_l) = 2$, we have $S_{xy_l} = \{f''_1(y_l, y'_l)\} = \{\gamma\}$. Recall that by $Assumption$ 5.9, $\alpha \in C' - F'_x(c_1)$ and thus $\alpha \in C' - F'_x(f''_1)$. Clearly $\alpha \neq \eta$. Recolor the edge xy_j with color α to get a coloring f'''_1. Note that the color α is valid for the edge xy_j because if there is a (α, η) bichromatic cycle, then it implies that $S_{xy_l} = \{\alpha\}$. But we know that $S_{xy_l} = \{\gamma\}$, a contradiction. Thus let $h_1 = f'''_1$ and the situation reduces to having only one bichromatic cycle (i.e., $|C_2| = 1$ and $|C_1| = 0$) □

In view of $Claim$ 5.22, color 2 is a candidate for the edge $y_j y'_j$. Recolor edge $y_j y'_j$ with color 2 to get a coloring f''_1. If the color 2 is valid for the edge $y_j y'_j$, then let $h_1 = f''_1$ and the situation reduces to having only one bichromatic cycle (i.e., $|C_2| = 1$ and $|C_1| = 0$). If the color 2 is not valid for the edge $y_j y'_j$, then there has to be a $(\gamma, 2)$ bichromatic cycle created due to the recoloring, thereby reducing the situation to $|C_2| = 2$ and $|C_1| = 0$. Now we can recolor the graph using the procedure similar to that in the proof of $Claim$ 5.18 (i.e., derangement of colors in C_2) to get a valid coloring h_1 without any bichromatic cycles.

The coloring f'_1 satisfied $Property\ A$ and hence none of the $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall \gamma \in C - \{1, 2\}$, contained the vertex y_j. Thus none of the $(\gamma, 1, q, y_1)$ maximal bichromatic paths will be $broken$ or $curtailed$ in the process of getting h_1 from f'_1. This is because we are changing only the colors of the edges incident on the vertex y_j or y_k and if a $(\gamma, 1, q, y_1)$ maximal bichromatic path gets $broken$ or $curtailed$, it means that the vertex y_j or y_k was contained in those maximal bichromatic path, a contradiction to $Property\ A$ of f'_1 since $y_j \in N'_G(x) - \{y_1\}$. On the other hand, if any of these paths gets extended, then vertex $y'_j \in \{y_1, q\}$. But in view of $Lemma$ 5.15 (part (a)) this is not possible. Thus the $(\gamma, 1, q, y_1)$ maximal bichromatic paths have not been extended. Thus these bichromatic paths have not been altered by the recolorings to get h_1 from f'_1. Thus the coloring h_1 satisfies $Property\ A$.

■

Observation 5.23. *Note that the color of the edge y_1y_1' is unaltered in h_1, i.e., $h_1(y_1, y_1') = f_1'(y_1, y_1') = 1$ (by $Observation$ 5.19). Also only the colors of certain edges incident on the vertex y_i, where $y_i \in N_G'(x) - \{y_1\}$ are modified.*

It is easy to see that h_1 is proper but not valid. It is not valid because, if it is valid then since h_1 satisfies $Property\ A$, there are $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall\gamma \in C - \{1, 2\}$. Thus by $Fact$ 2.1, for any $\theta \in C - F_x(h_1)$, there cannot be a $(1, \theta, xy_1)$ critical path. Thus by $Fact$ 2.5, color θ is valid for the edge xy_1. Thus we have a valid coloring for the graph G, a contradiction. Thus h_1 is not valid. Then in view of $Lemma$ 5.20, we make the following assumption:

Assumption 5.24. *Without loss of generality let the only bichromatic cycle in the coloring h_1 of G_1 pass through the vertex y_j, $j \neq 1$. Also let $h_1(x, y_j) = \rho$.*

We get a coloring c_j of G_j from h_1 of G_1 by:

1. Removing the edge xy_j.

2. Adding the edge xy_1 and coloring it with the color $h_1(x, y_j) = \rho$.

Note that the coloring c_j is proper since $\rho \neq c_j(y_1, y_1') = h_1(y_1, y_1') = 1$ (by $Observation$ 5.23) and $\rho \notin S_{y_1x}(c_j)$ (by the definition of c_j). Note that by removing the edge xy_j we have broken the only bichromatic cycle that existed with respect to h_1. The coloring c_j is valid because if there is a bichromatic cycle in G_j with respect to c_j then it should contain the edge xy_1 and thus it should be a $(1, \rho)$ bichromatic cycle since $c_j(x, y_1) = h_1(x, y_j) = \rho$ and $c_j(y_1, y_1') = 1$. The $(\rho, 1, q, y_1)$ maximal bichromatic path with respect to h_1 is still a bichromatic path with respect to c_j. And since no edge incident to q is recolored, there is a $(\rho, 1)$ maximal bichromatic path that starts at q and contains vertex y_1. This clearly implies that there cannot be a $(\rho, 1)$ bichromatic cycle containing vertex y_1. It follows that the coloring c_j of G_j is acyclic. Therefore all the Lemmas in previous sections are applicable to the coloring c_j also.

Now we may assume that $c_j(y_j, y'_j) = 2$ because if $c_j(y_j, y'_j) = 1$, then we can change the color of the edge $y_j y'_j$ to 2 without altering the validity of the coloring since y_j is a pendant vertex in G_j. Thus we make the following assumption:

Assumption 5.25. *Without loss of generality let* $c_j(y_j, y'_j) = 2$. *Also recall that* $c_j(x, q) = 2$ *and* $c_j(x, q') = 1$.

Remark: In view of $Assumption$ 5.25, the $Critical\ Path\ Property$ with respect to the coloring c_j of G_j reads as follows: With respect to the coloring $c_{j,\gamma}$, there exists a $(2, \gamma, xy_j)$ critical path, for all $\gamma \in C'$. The reader may contrast the $Critical\ Path\ Property$ of c_j with that of c_1 (See remark after $Assumption$ 5.16). This correspondence is very important for the proof.

Observation 5.26. *Note that* $c_j(x, q) = 2$, $c_j(x, q') = 1$, $c_j(y_1, y'_1) = 1$, $c_j(y_j, y'_j) = 2$ *and* $c_j(x, y_1) = \rho \notin \{1, 2\}$. *Also if* e *is an edge such that none of its end points is* x *or* y_i, *where* $y_i \in N'_G(x)$, *we have* $c_j(e) = c_1(e)$.

Lemma 5.27. *Coloring* $c_{j,\gamma}$ *of* $G_{j,\gamma}$ *satisfies* $Property\ A$.

Proof: We consider the following cases:

case 1: $\gamma \in C' - \{\rho\}$

Recall that the coloring h_1 satisfied $Property\ A$. In getting c_j from h_1, we have only colored the edge xy_1 with color ρ and have discarded the edge xy_j. Thus $\forall \gamma \in C' - \{\rho\}$, there exists a $(\gamma, 1, q, y_1)$ maximal bichromatic path in c_j also. Noting that by $Property\ A$, the maximal bichromatic path does not contain vertex x or y_i, where $\forall y_i \in N'_G(x) - \{y_1\}$, we infer that even in $G_{j,\gamma}$ the $(\gamma, 1, q, y_1)$ maximal bichromatic path is unaltered.

case 2: $\gamma = \rho$

Then $G_{j,\rho}$ is the graph obtained by removing the edge xy_1 from G_j since $c_j(x, y_1) = \rho$. Recall that with respect to the coloring h_1 we have a $(\rho, 1, q, y_1)$ maximal bichromatic path. Removal of edge xy_j from G_1 cannot alter this path since h_1 satisfies $Property\ A$ and thus edge xy_j is not in the path. Now the graph obtained is nothing but the graph $G_{j,\rho}$ with respect to the coloring $c_{j,\rho}$. Thus $G_{j,\rho}$ satisfies $Property\ A$. ■

Property B: Let $c_{1,\eta}$ be a partial coloring of $G_{1,\eta}$, for $\eta \in C - \{1, 2\}$. Then $c_{1,\eta}$ is said to satisfy $Property\ B$ iff $\forall\gamma \in C - \{1,2\}$, there exists a $(\gamma, 2)$ maximal bichromatic path which starts at vertex q and involves the vertex y'_j. Also the length of the segment of this bichromatic path between the vertices q and y'_j is at least three. Moreover in none of the above maximal bichromatic paths the segment between the vertices q and y'_j contains vertex x or vertex y_i, where $y_i \in N'_G(x)$.

Lemma 5.28. *Coloring* $c_{1,\eta}$ *of* $G_{1,\eta}$ *satisfies* $Property\ B$*, for* $\eta \in C - \{1,2\}$.

Proof: By $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10) and $Lemma$ 5.14, $\forall\gamma \in C'$, there exists a $(2, \gamma, x, y_j)$ critical path of length at least five in $G_{j,\gamma}$. Also by $Lemma$ 5.13, these critical paths do not contain vertex y_i, $\forall y_i \in N'_G(x) - \{y_j\}$. Recall that we obtained c_j from c_1 by a series of recolorings. How will the above mentioned critical paths change if we undo all these recolorings and get back c_1? Note that in the process of obtaining coloring c_j from c_1, we have only changed the colors incident on the vertices y_i, where $y_i \in N'_G(x)$ and have exchanged the colors of the edges xq and xq' (by $Observation$ 5.26). Thus only the colors of edge xq and possibly edge $y_jy'_j$ of these critical paths will get modified when we undo the recolorings. The reader may recall that the first step in getting c_j from c_1 was to exchange the colors of edges xq and xq'. It follows that with respect to a coloring $c_{1,\eta}$, there exists a $(\gamma, 2)$ maximal bichromatic path which $starts\ at$ vertex q and involves the vertex y'_j. It also follows that the length of the segment of the bichromatic path between the vertices q and y'_j is at least three. Moreover it is easy to see that none of the above maximal bichromatic paths the segment between the vertices q and y'_j contains vertex x or vertex y_i, where $y_i \in N'_G(x)$. ■

5.2.4 Selection of secondary pivot p and properties of c_1 and c_j in the vicinity of p

Let $N'_G(q) = N_G(q) \cap (W_1 \cup W_0)$ and $N''_G(q) = N_G(q) - N'_G(q)$. Since $q \in W_2 \cup W_1$ (see $Assumption$ 5.5) it is easy to see that $|N''_G(q)| \leq 2$. Now recall that in view of (5.2) $degree_G(q) = \Delta$ and by (5.3), $\Delta \geq 4$. Thus we have $|N'_G(q)| \geq 2$.

Let $p \in N'_G(q)$ be such that $p \neq x$. In the rest of the proof, this vertex p will play a central role. Therefore we name it as the $Secondary\ Pivot$. Let $c_1(q,p) = \eta$. Note that $\eta \in C'$ by (5.1). Thus by $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10), there exists a

$(1, \eta, xy_1)$ critical path with respect to the coloring $c_{1,\eta}$ that passes through the vertex p and clearly qp is the second edge of this critical path. Recalling that this critical path has length at least five (by $Lemma$ 5.14), we can infer that $p \neq y_1$ and $degree_{G_1}(p) \geq 2$. Now since $p \in W_1 \cup W_0$, there is at most one neighbor of p other than q which is not in W_0. If such a vertex exists let it be p'. Otherwise clearly $(N_G(p) \cap W_0) \neq \emptyset$ and let $p' \in N_G(p) \cap W_0$. Thus $N_G(p) - \{q, p'\} \subseteq W_0$. If $N_G(p) - \{q, p'\} \neq \emptyset$, let $N_G(p) - \{q, p'\} = \{z_1, z_2, \ldots, z_k\}$. Also $\forall z_i$, let $N_G(z_i) = \{p, z'_i\}$ (See $figure$ 5.3) (At this point the reader may note that the primary pivot x and secondary pivot p are somewhat structurally similar).

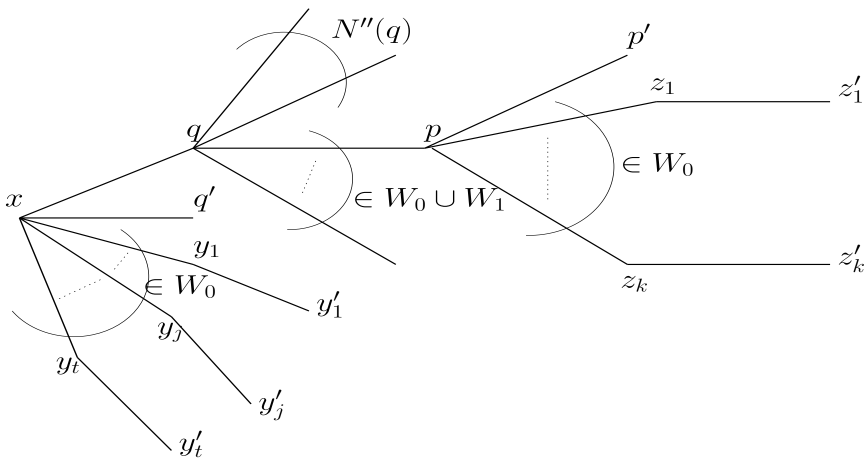

Figure 5.3: Vertex p and its neighbors

Lemma 5.29. $x, y_i \notin \{p, p', z_1, \ldots, z_k, z'_1, \ldots, z'_k\}$, *for* $y_i \in N'_G(x)$.

Proof: First note that $x \neq p$, by the definition of p. It is easy to see that $x \notin \{p', z_1, \ldots, z_k\}$, by part (a) of $Lemma$ 5.15. Now $x \notin \{z'_1, \ldots, z'_k\}$ because otherwise z_i will be some y_i and hence $p = y'_i$, But now there is an edge between q and p, a contradiction to part (b) of $Lemma$ 5.15. Similarly from part (a) of $Lemma$ 5.15, $y_i \neq p$ and from part (b) of $Lemma$ 5.15, $y_i \notin \{p', z_1, \ldots, z_k\}$. Now if $y_i \in \{z'_1, \ldots, z'_k\}$, then since $x \neq z_i$, we have $y'_i = z_i$, a contradiction since $degree_G(y'_i) = \Delta \geq 4$ (by (5.2)) and $degree_G(z_i) = 2$. Thus $x, y_i \notin \{p, p', z_1, \ldots, z_k, z'_1, \ldots, z'_k\}$, for $y_i \in N'_G(x)$. ■

Lemma 5.30. $c_j(q, p) = c_1(q, p) = \eta$.

Proof: Recall that by $Observation$ 5.26, only the edges incident on vertices x or y_i, where $y_i \in N'_G(x)$ are altered while obtaining coloring c_j from c_1. Now to show that $c_j(q, p) =$

$c_1(q,p) = \eta$, its enough to verify that $q, p \notin \{x\} \cup N'_G(x)$. But this is obvious from part (a) of $Lemma$ 5.15.

■

Lemma 5.31. $\{1,2\} \subseteq S_{qp}(c_{1,\eta})$.

Proof: By $Lemma$ 5.14, we know that $1 \in S_{qp}(c_{1,\eta})$ since qp is only the second edge of the $(1, \eta, xy_1)$ critical path which is guaranteed to have length at least five with respect to the coloring $c_{1,\eta}$ of $G_{1,\eta}$. Now by $Lemma$ 5.28, with respect to $c_{1,\eta}$ there exists a $(\eta, 2)$ maximal bichromatic path which starts at vertex q and contains vertex y'_j. Moreover the segment of the bichromatic path between the vertices q and y'_j is of length at least three with respect to $c_{1,\eta}$. Since this $(\eta, 2)$ maximal bichromatic path starts with edge qp colored η, we infer that $2 \in S_{qp}(c_{1,\eta})$. Thus $\{1,2\} \subseteq S_{qp}(c_{1,\eta})$.

■

Remark: In view of $Lemma$ 5.31, $degree_G \geq 3$. Therefore $p \notin W_0$. It follows that $p \in W_1$. It is interesting to note that p could have been selected as the $Primary\ Pivot$ instead of x. The reader may want to reread the procedure for selecting the primary pivot given at the beginning of Section 3. With respect to this procedure vertex p is symmetric to vertex x and thus is an equally eligible candidate to be the primary pivot. It follows that the structure of the minimum counter example at the vicinity of p is symmetric to the structure at the vicinity of x. More specifically we have the following Lemma, corresponding to $Lemma$ 5.15:

Lemma 5.32. *The minimum counter example G satisfies the following properties,*

(a) $\forall u, v \in N_G(p)$, $(u,v) \notin E(G)$.

(b) $\forall z_i \in N_G(p) - \{q, p'\}$ *and* $\forall v \in N_G(p)$, *we have* $(v, z'_i) \notin E(G)$.

This Lemma is not explicitly used in the proof, but we believe that this information will help the reader to visualize the situation better.

In view of $Lemma$ 5.31 let e_1 and e_2 be the edges incident on p such that $c_{1,\eta}(e_1) = 1$ and $c_{1,\eta}(e_2) = 2$. Then we claim the following:

Lemma 5.33. $c_{j,\eta}(e_1) = 1$ *and* $c_{j,\eta}(e_2) = 2$.

Proof: Recall that by $Observation$ 5.26, only the edges incident on vertices x or y_i, where $y_i \in N'_G(x)$ are altered while obtaining coloring c_j from c_1. Let $e_1 = (p, z_{i_1})$ and $e_2 = (p, z_{i_2})$. Now to show that $c_{j,\eta}(e_1) = 1$ and $c_{j,\eta}(e_2) = 2$, it is enough to verify that $p, z_{i_1}, z_{i_2} \notin \{x\} \cup N'_G(x)$. But this true by $Lemma$ 5.29.

■

Lemma 5.34. $c_{1,\eta}(p, p') \in \{1, 2\}$ (*In other words, one of the edge* e_1 *or* e_2 *is* pp'. *By* $Lemma$ *5.33, this also implies that* $c_{j,\eta}(p, p') = c_{1,\eta}(p, p') \in \{1, 2\}$).

Proof: Suppose not. Then $e_1 \neq pp'$ and $e_2 \neq pp'$. Without loss of generality let $e_1 = (p, z_1)$ and $e_2 = (p, z_2)$. Thus $c_{1,\eta}(p, z_1) = 1$ and $c_{1,\eta}(p, z_2) = 2$. By $Lemma$ 5.14 there exists a $(1, \eta, xy_1)$ critical path of length at least five with respect to $c_{1,\eta}$. This implies that $c_{1,\eta}(z_1, z'_1) = \eta$. Now by $Lemma$ 5.28, with respect to $c_{1,\eta}$ there exists a $(\eta, 2)$ maximal bichromatic path which starts at vertex q and contains vertex y'_j. Moreover the segment of this bichromatic path between the vertices q and y'_j is of length at least three with respect to $c_{1,\eta}$. Since pz_1 is only the second edge of this path, we can infer that $c_{1,\eta}(z_2, z'_2) = \eta$.

Now with respect to the coloring $c_{1,\eta}$, we exchange the colors of the edges pz_1 and pz_2 to get a coloring $c'_{1,\eta}$.

Claim 5.35. *Coloring* $c'_{1,\eta}$ *is valid.*

Proof: Note that $c'_{1,\eta}$ is proper since $c_{1,\eta}(z_1, z'_1) = \eta$ and $c_{1,\eta}(z_2, z'_2) = \eta$. Now the coloring $c'_{1,\eta}$ is valid because otherwise there has to be a $(\eta, 1)$ or $(\eta, 2)$ bichromatic cycle since only the colors of the edges pz_1 and pz_2 are altered. Thus such a bichromatic cycle has to contain the edge qp since $c'_{1,\eta}(q, p) = \eta$. From (5.1), we can infer that color $2 \notin F_q(c'_{1,\eta})$. But if there exists a bichromatic cycle with respect to the coloring $c'_{1,\eta}$, it has to contain vertex q. From this we can infer that it has to be a $(\eta, 1)$ bichromatic cycle. This means that the cycle has to contain the vertex x since $c'_{1,\eta}(x, q) = 1$. But we know by definition of $c_{1,\eta}$ that $\eta \notin F_x(c_{1,\eta}) = F_x(c'_{1,\eta})$. Thus there does not exist a $(\eta, 1)$ bichromatic cycle with respect to the coloring $c'_{1,\eta}$. We conclude that the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ is valid. □

Claim 5.36. *With respect to the partial coloring* $c'_{1,\eta}$, *there does not exist any* $(1, \eta, xy_1)$ *critical path.*

Proof: Now since $c'_{1,\eta}$ is proper, taking $u = p$, $i = z_1$, $j = z_2$, $ab = xy_1$, $\lambda = 1$, $\xi = \eta$ and noting that $\{x, y_1\} \cap \{z_1, z_2\} = \emptyset$ (by $Lemma$ 5.29), $Lemma$ 2.8 can be applied. There existed a $(1, \eta, xy_1)$ critical path containing vertex p in coloring $c_{1,\eta}$. By $Lemma$ 2.8, we infer that there cannot be any $(1, \eta, xy_1)$ critical path in the coloring $c'_{1,\eta}$. $\square$

Claim 5.37. *There exists a valid coloring c'_1 of G_1 such that the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ is derivable from c'_1.*

Proof: It is enough to show that we can extend the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ to a valid coloring c'_1 of G_1. If $\eta \in C - F_x(c_1)$, then by definition $G_{1,\eta} = G_1$ and thus $c'_1 = c'_{1,\eta}$. Otherwise let $y_k \in N'_G(x)$ be the vertex such that $c_1(x, y_k) = \eta$. Note that $k \neq 1$. Recall that $c_{1,\eta}$ is obtained by discarding the color on the edge xy_k. Thus it is enough to extend the coloring $c'_{1,\eta}$ to c'_1 by assigning an appropriate color to the edge xy_k.

Note that there exists a $(1, \alpha, xy_1)$ critical path with respect to $c_{1,\eta}$, for $\alpha \in C - F_x(c_1)$ (by $Lemma$ 5.7). Clearly $\alpha \neq \eta$. We claim that the $(1, \alpha, xy_1)$ critical path exists even with respect to $c'_{1,\eta}$. To see this note that we have changed the colors of only edges pz_1 and pz_2 to get $c'_{1,\eta}$ from $c_{1,\eta}$. Note that by this exchange the $(1, \alpha, xy_1)$ critical path cannot be extended since $p, z_2 \notin \{x, y_1\}$ (by $Lemma$ 5.29). Now if the $(1, \alpha, xy_1)$ critical path gets altered it means that this critical path contained the edge pz_1 (recall that $c_{1,\eta}(p, z_1) = 1$) and hence $c_{1,\eta}(z_1, z'_1) = \alpha$. But we know that $c_{1,\eta}(z_1, z'_1) = \eta$, a contradiction. Thus we have,

$$\text{\textit{With respect to the partial coloring } } c'_{1,\eta}\text{\textit{, there exists a} } (1, \alpha, xy_1) \text{ \textit{critical path,}} \\ \text{\textit{for} } \alpha \notin F_x(c'_{1,\eta}) \text{ \textit{and} } \alpha \neq \eta. \tag{5.5}$$

Now color the edge xy_k with color η to get a coloring d_1 of G_1. If d_1 is valid we are done and $c'_1 = d_1$. If it is not valid, then there has to be a bichromatic cycle containing the color η. Note that the coloring d_1 and c_1 differ only due to the exchange of colors of edges pz_1 and pz_2. Thus it has contain one of the edges pz_1 or pz_2. Therefore it has to be either a $(\eta, 1)$ or $(\eta, 2)$ bichromatic cycle since $d_1(p, z_1) = 2$, $d_1(p, z_2) = 1$. This also means that the bichromatic cycle has to contain the vertex q, since $d(p, q) = \eta$. Thus the bichromatic cycle has to be a $(\eta, 1)$ bichromatic cycle since $2 \notin F_q(d_1)$. This means that $d_1(y_k, y'_k) = 1$. Now recolor the edge xy_k with color α to get a coloring d'_1 of G_1. If d'_1 is valid we are done and $c'_1 = d'_1$. If it is not valid then there has to be a $(\alpha, 1)$ bichromatic cycle containing the vertex x, implying that there existed a $(1, \alpha, xy_k)$ critical path with respect to the coloring d_1 and hence with respect to the coloring $c'_{1,\eta}$. But in view of (5.5), there already exists a $(1, \alpha, xy_1)$ critical path and by $Fact$ 2.1, $(1, \alpha, xy_k)$ critical path is not possible, a contradiction. Thus the coloring d'_1 is valid and let $c'_1 = d'_1$.

Thus there exists a valid coloring c'_1 of G_1 such that the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ is derivable from c'_1. □

Now in view of $Claim$ 5.36 and $Claim$ 5.37 there does not exist any $(1, \eta, xy_1)$ critical path with respect to the coloring $c'_{1,\eta}$ of $G_{1,\eta}$, a contradiction to $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10).

We conclude that $c_{1,\eta}(p, p') \in \{1, 2\}$.

■

Assumption 5.38. *In view of $Lemma$ 5.31, $Lemma$ 5.33 and $Lemma$ 5.34, let z_1 be the vertex such that $\{c_{1,\eta}(p, z_1)\} = \{1, 2\} - \{c_{1,\eta}(p, p')\}$. It follows that $\{c_{j,\eta}(p, z_1)\} = \{1, 2\} - \{c_{j,\eta}(p, p')\}$ and $\{e_1, e_2\} = \{pp', pz_1\}$.*

Observation 5.39.

(a) *If $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 2$, we have by Assumption 5.38, that $c_{1,\eta}(p, z_1)$=$c_{j,\eta}(p, z_1) = 1$. Thus with respect to the partial coloring $c_{1,\eta}$, there exists a $(1, \eta, xy_1)$ critical path of length at least five which contains the vertex z_1. It follows that $c_{1,\eta}(z_1, z'_1) = \eta$ since $z_1z'_1$ is just the fourth edge of this $(1, \eta, xy_1)$ critical path.*

(b) *If $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 1$, we have by Assumption 5.38, that $c_{1,\eta}(p, z_1)$=$c_{j,\eta}(p, z_1) = 2$. Thus with respect to the partial coloring $c_{j,\eta}$, there exists a $(2, \eta, xy_j)$ critical path of length at least five which contains the vertex z_1. It follows that $c_{j,\eta}(z_1, z'_1) = \eta$ since $z_1z'_1$ is just the fourth edge of this $(2, \eta, xy_j)$ critical path.*

Local Recolorings: If a partial coloring h of G is obtained from a partial coloring c of G by recoloring only certain edges incident on the vertices belonging to $N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\}$ and also possibly the edge pp', then h is said to be obtained from c by local recolorings.

The concept of local recolorings turns out to be crucial for the rest of the proof. The following lemma provides the main tool in this respect.

Lemma 5.40.

(a) *Let* $c_{1,\eta}(p,p') = c_{j,\eta}(p,p') = 2$*. Also let* $h_{1,\eta}$ *be any valid coloring obtained from* $c_{1,\eta}$ *by recoloring only certain edges incident on the vertices belonging to* $N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\}$ *and also possibly the edge* pp' *(i.e., by only local recolorings). Then there exists a valid coloring* h_1 *of* G_1 *such that the valid coloring* $h_{1,\eta}$ *of* $G_{1,\eta}$ *is derivable from* h_1*.*

(b) *Let* $c_{1,\eta}(p,p') = c_{j,\eta}(p,p') = 1$*. Also let* $f_{j,\eta}$ *be any valid coloring obtained from* $c_{j,\eta}$ *by recoloring only certain edges incident on the vertices belonging to* $N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\}$ *and also possibly the edge* pp' *(i.e., by only local recolorings). Then there exists a valid coloring* f_j *of* G_j *such that the valid coloring* $f_{j,\eta}$ *of* $G_{j,\eta}$ *is derivable from* f_j*.*

Proof:

(a) Recall that $\eta \neq 1, 2$. If $\eta \notin F_x(c_1)$, then $c_{1,\eta} = c_1$. In this case we take $h_1 = c_1$. Otherwise if $\eta \in F'_x(c_1)$, let xy_k be the edge in G_1 such that $c_1(x, y_k) = \eta$. Note that $k \neq 1$. It is enough to show that we can extend the valid coloring $h_{1,\eta}$ of $G_{1,\eta}$ to a valid coloring h_1 of G_1 by assigning an appropriate color to the edge xy_k (Reader may note that neither pp' nor any edge incident on the vertices in $\{z_1, z_2, \ldots, z_k\}$ can be the edge xy_k since $x \notin \{p, p', z_1, z_2, \ldots, z_k, z'_1, z'_2, \ldots, z'_k\}$ due to *Lemma* 5.29). Now assign color η to the edge xy_k to get a coloring d. If the coloring d is valid we are done and we have $h_1 = d$. If it is not valid then there has to be a bichromatic cycle created in G_1 with respect to the coloring d. The cycle has to be a (η, θ) bichromatic cycle, where $d(y_k, y'_k) = \theta$. Moreover we can infer that $\theta \in F_x(d)$. If $\theta \neq d(p, p') = 2$, then let $d' = d$. Otherwise we have $\theta = d(p, p') = 2$. Now there exists a color $\omega \neq 2, \eta$ that is a candidate for the edge $y_k y'_k$. Recolor the edge $y_k y'_k$ using color ω to get a coloring d' of G_1. Now if d' is a valid coloring, then we are done and we have $h_1 = d'$. If it is not valid, then $d'(y_k, y'_k) \neq 2$. Let $d'(y_k, y'_k) = \beta \neq 2$. Moreover with respect to the coloring d' there should be a (η, β) bichromatic cycle. Also let $\alpha\ (\neq \eta)\ \in C - F_x(c_1) = C - F_x(d')$. Now if,

(1) $\beta = 1$.

Claim 5.41. *None of the* $(1, \gamma, xy_1)$ *critical paths, where* $\gamma\ (\neq \eta)\ \in C - F_x(c_1)$ *are altered in the process of getting the coloring* $h_{1,\eta}$ *from* $c_{1,\eta}$*.*

Proof: Recall that only the edges incident on vertices z_i, where $z_i \in N_G(p) - \{p', q\}$ and edge pp' are possibly recolored to get the coloring $h_{1,\eta}$ of $G_{1,\eta}$ from

$c_{1,\eta}$. Note that by these recolorings the $(1, \gamma, xy_1)$ critical path cannot be extended since $x, y_1 \notin \{p, p', z_1, z_2, \ldots, z_k, z'_1, z'_2, \ldots, z'_k\}$ due to $Lemma$ 5.29. Now if any $(1, \gamma, xy_1)$ critical paths are altered then they have to contain the above mentioned edges. Note that none of the vertices in $\{z_1, z_2, \ldots, z_k\}$ or vertex p' can be the end vertices x or y_1 and hence any critical path containing the vertex z_i or p' should also contain the vertex p since $degree_{G_{1,\eta}}(z_i) = 2$. We can infer that with respect to the coloring $c_{1,\eta}$, the $(1, \gamma, xy_1)$ critical path passes through the vertex p. It follows that this critical path has to contain the edge pz_1 since $c_{1,\eta}(p, z_1) = 1$ (from part (a) of $Observation$ 5.39). Now since $z_1 \in W_0$ (i.e., $degree_{G_{1,\eta}}(z_1) = 2$), this implies that $c_{1,\eta}(z_1, z'_1) = \gamma$, a contradiction since from part (a) of $Observation$ 5.39, we know that $c_{1,\eta}(z_1, z'_1) = \eta$. Thus there cannot be any $(1, \gamma, xy_1)$ critical path containing the edges incident on vertices z_i, where $z_i \in N_G(p) - \{p', q\}$ and edge pp'. Thus none of the $(1, \gamma, xy_1)$ critical paths, where $\gamma \in C - F_x(c_1)$, $\gamma \neq \eta$ are altered. □

Since d' is not valid there has to be a $(\eta, 1)$ bichromatic cycle that passes through the vertex x. Now recolor the edge xy_k with color α to get a coloring d''. Now if still there is a bichromatic cycle, then it should contain the edge xy_k and hence the edge $y_k y'_k$. Therefore it is a $(\alpha, 1)$ bichromatic cycle. This implies by $Fact$ 2.5 that there existed a $(1, \alpha, xy_k)$ critical path with respect to the coloring d' and hence with respect to the coloring $h_{1,\eta}$. But in view of $Claim$ 5.41, there exists a $(1, \alpha, xy_1)$ critical path with respect to the coloring $h_{1,\eta}$, a contradiction in view of $Fact$ 2.1. Thus the coloring d'' is valid.

(2) $\beta \neq 1$. This implies that $\beta\ (\neq \eta) \in F'_x(d')$. Let $y_t \in N'_G(x)$ be such that $d'(x, y_t) = \beta$. Thus $d'(y_t, y'_t) = \eta$. Now recolor the edge xy_k with color $\alpha \in C - F_x(d')$ to get a coloring d''. Note that $\alpha \neq \eta$ since $\eta \notin C - F_x(d')$. Now if still there is a bichromatic cycle, then it should contain the edge xy_k and hence the edge $y_k y'_k$. Therefore it is a (α, β) bichromatic cycle. Thus the bichromatic cycle should contain the edge xy_t. Since $degree_{G_1}(y_t) = 2$, the bichromatic cycle should contain the edge $y_t y'_t$. But by our assumption, $d''(y_t, y'_t) = d'(y_t, y'_t) = \eta \neq \alpha$, a contradiction. Thus the coloring d'' is valid.

Now let $h_1 = d''$. Thus we get a valid coloring of G_1 from $h_{1,\eta}$.

(b) The proof of this is similar to that of part (a) with G_j, c_j, y_j taking the roles of G_1, c_1, y_1 respectively and the colors 1 and 2 exchanging their roles.

■

Lemma 5.42.

(a) If $c_{1,\eta}(p,p') = c_{j,\eta}(p,p') = 2$, then with respect to the coloring $c_{1,\eta}$, $2 \notin S_{z_1z_1'}$. (Recall that by Assumption 5.38, $\{c_{1,\eta}(p,z_1)\} = \{c_{j,\eta}(p,z_1)\} = \{1,2\} - \{c_{1,\eta}(p,p')\} = \{1\}$.)

(b) If $c_{j,\eta}(p,p') = c_{1,\eta}(p,p') = 1$, then with respect to the coloring $c_{j,\eta}$, $1 \notin S_{z_1z_1'}$. (Recall that by Assumption 5.38, $\{c_{j,\eta}(p,z_1)\} = \{c_{1,\eta}(p,z_1)\} = \{1,2\} - \{c_{j,\eta}(p,p')\} = \{2\}$.)

Proof:

(a) Suppose not. That is $2 \in S_{z_1z_1'}$. Note that by part (a) of $Observation$ 5.39, we have $c_{j,\eta}(p,z_1) = 1$ and $c_{j,\eta}(z_1,z_1') = \eta$. Therefore there exists some $\theta \notin \{1,2,\eta\}$ missing in $S_{z_1z_1'}$. Now recolor edge z_1z_1' with color θ to get a coloring $c'_{1,\eta}$. If the coloring $c'_{1,\eta}$ is valid, then let $c''_{1,\eta} = c'_{1,\eta}$. Otherwise a bichromatic cycle gets formed by the recoloring. Since $c'_{1,\eta}(p,z_1) = 1$, it has to be a $(1,\theta)$ bichromatic cycle and it passes through the vertex p. Thus there exists $z_i \in N_G(p) - \{q,p'\}$ such that $c'_{1,\eta}(p,z_i) = \theta$ and $c'_{1,\eta}(z_i,z_i') = 1$.

Now there exists a color $\mu \notin \{1,\theta,2,\eta\}$ missing at p. Recolor the edge pz_1 with color μ to get a coloring $c''_{1,\eta}$. This clearly breaks the $(1,\theta)$ bichromatic cycle that existed with respect to $c'_{1,\eta}$. But if a new bichromatic cycle gets formed with respect to $c''_{1,\eta}$, then it has to contain vertex z_1 and therefore the edge z_1z_1', implying that it has to be a (μ,θ) bichromatic cycle since $c''_{1,\eta}(z_1,z_1') = \theta$. This cycle passes through the vertex p and hence passes through the vertex z_i since $c''_{1,\eta}(p,z_i) = \theta$, implying that $c''_{1,\eta}(z_i,z_i') = \mu$, a contradiction since $c''_{1,\eta}(z_i,z_i') = 1$. Thus the coloring $c''_{1,\eta}$ is valid.

Note that we have possibly changed the colors of the edges pz_1 and z_1z_1' to get $c''_{1,\eta}$ from $c_{1,\eta}$ (i.e., only local recolorings are done). Therefore by part (a) of $Lemma$ 5.40 we infer that there exists a coloring c''_1 of G_1 such that $c''_{1,\eta}$ is derivable from c''_1. *It follows from $Critical\ Path\ Property$ (i.e., $Lemma$ 5.7 or $Lemma$ 5.10) that there exists a $(1,\eta,xy_1)$ critical path with respect to the coloring $c''_{1,\eta}$.* On the other hand recall that with respect to $c_{1,\eta}$ there existed a $(1,\eta,xy_1)$ critical path passing through pz_1 and z_1z_1' (by part (a) of $Observation$ 5.39). But while getting $c''_{1,\eta}$ from $c_{1,\eta}$ we have indeed changed the color of at least one of the edges pz_1 or z_1z_1' using a color other than 1 and η. It follows that the $(1, \eta)$ maximal bichromatic path which contains the vertex x ends at either vertex p or z_1. Noting that $p, z_1 \neq y_1$, we infer by $Fact$ 2.1 that there cannot be a $(1, \eta, xy_1)$ critical path with respect to the coloring $c''_{1,\eta}$, a contradiction.

(b) The proof of this is similar to that of part (a) with $G_{j,\eta}$, $c_{j,\eta}$, y_j taking the roles of $G_{1,\eta}$, $c_{1,\eta}$ and y_1 respectively and the colors 1 and 2 exchanging their roles.

■

5.2.5 Getting a valid coloring that contradicts the Critical Path Property either from c_1 or from c_j

In this section we will get the final contradiction in the following way: If $c_{1,\eta}(p, p')$ = $c_{j,\eta}(p, p')$ = 1, then we will show that we can get a coloring c_j' from c_j that contradicts the Critical Path Property. Otherwise if $c_{1,\eta}(p, p')$ = $c_{j,\eta}(p, p') = 2$, then we will show that we can get a coloring c_1' from c_1 that contradicts the $Critical\ Path\ Property$.

The two colorings c_1 and c_j are very similar and hence we will only describe the way we get c_1' from c_1. The same arguments can be imitated easily for c_j by keeping the following correspondences in mind.

1. Vertex y_1 has same role as vertex y_j.
2. Colors 1 and 2 exchange their roles.
3. $(1, \gamma, xy_1)$ critical path has the same role as $(2, \gamma, xy_j)$ critical path, for $\gamma \in C'$. The $Critical\ Path\ Property$ of c_1 corresponds to that of c_j (See Remarks after $Assumption$ 5.16 and $Assumption$ 5.25).
4. Part (a) of $Lemma$ 5.40 and $Lemma$ 5.42 applies to coloring c_1 while part (b) applies to coloring c_j in a corresponding way.
5. $Lemma$ 5.28 has the same role as $Lemma$ 5.27.

We make the following assumption:

Assumption 5.43. *Let* $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 2$.

Observation 5.44. *In view of* $Assumption$ *5.43 and* $Observation$ *5.39, there exists* $(1, \eta, xy_1)$ *critical path which contains the vertex* z_1 *with respect to the partial coloring* $c_{1,\eta}$*. Moreover this path is of length at least five. It follows that* $c_{1,\eta}(p, z_1) = 1$ *and* $c_{1,\eta}(z_1, z_1') = \eta$*. The first five vertices of the path are* x*,* q*,* p*,* z_1*,* z_1'*. Then clearly* $z_1' \neq y_1$ *and hence is not a pendant vertex in* $G_{1,\eta}$*. Thus we have* $S_{z_1 z_1'} \neq \emptyset$ *and* $1 \in S_{z_1 z_1'}$*.*

Getting a valid coloring d_1 of $G_{1,\eta} - \{pz_1\}$ from $c_{1,\eta}$ by only local recolorings

In view of $Lemma$ 5.42 and since $c_{1,\eta}(p, z_1) = 1$, the color 2 is a candidate for the edge $z_1 z_1'$. We get a valid coloring d_1 of $G_{1,\eta} - \{pz_1\}$ from $c_{1,\eta}$ by removing the edge pz_1 and recoloring the edge $z_1 z_1'$ by the color 2. Note that d_1 is valid since z_1 is a pendant vertex in $G_{1,\eta} - \{pz_1\}$. Moreover we have broken the $(1, \eta, xy_1)$ critical path. Hence we have,

With respect to the partial coloring d_1*, there does not exists any* $(1, \eta, xy_1)$ *critical path.* (5.6)

Lemma 5.45. *With respect to the partial coloring* d_1 *of* $G_{1,\eta}$ *,* $\forall \gamma \in C - F_p(d_1)$*, there exists a* $(2, \gamma, pz_1)$ *critical path. Since each of these critical paths has to contain the edge* pp'*, we can infer that* $C - F_p(d_1) \subseteq S_{pp'}$*.*

Proof: Suppose not. Then there exists a color $\gamma \in C - F_p(d_1)$ such that there is no $(2, \gamma, pz_1)$ critical path. By $Fact$ 2.5 color γ is valid for the edge pz_1. Thus we get a valid coloring d_1' of $G_{1,\eta}$ by coloring the edge pz_1 with color γ.

Note that we have possibly changed the colors of the edges pz_1 and $z_1 z_1'$ to get d_1' from $c_{1,\eta}$ (i.e., only local recolorings are done). Therefore by part (a) of $Lemma$ 5.40 we infer that there exists a valid coloring of G_1 from which d_1' can be derived. *It follows from* $Critical\ Path\ Property$ *(i.e.,* $Lemma$ *5.7 or* $Lemma$ *5.10) that there exists a* $(1, \eta, xy_1)$ *critical path with respect to the coloring* d_1'. On the other hand recall that with respect to $c_{1,\eta}$ there existed a $(1, \eta, xy_1)$ critical path passing through pz_1 and $z_1 z_1'$ (by $Observation$ 5.44). But while getting d_1' from $c_{1,\eta}$ we have indeed changed the color of the edges $z_1 z_1'$ using the color $2 \notin \{1, \eta\}$. It follows that the $(1, \eta)$ maximal bichromatic path which contains the vertex x ends at either vertex p or z_1. Noting that $p, z_1 \neq y_1$, we infer that there cannot be a $(1, \eta, xy_1)$ critical path with respect to the coloring d_1', a contradiction.

■

Note that with respect to $G_{1,\eta} - \{pz_1\}$, $|F_p(d_1)| \leq \Delta - 1$ and therefore $|C - F_p(d_1)| \geq 2$. But we know that color $1 \notin F_p(d_1)$. Since $|C - F_p(d_1)| \geq 2$, there exists a color $\mu \neq 1 \in C - F_p(d_1)$. Note that $\mu \neq 2$ also. The following observation is obvious in view of $Claim$ 5.45:

Observation 5.46. *With respect to the partial coloring d_1 of $G_{1,\eta}$, $1, \mu \notin F_p(d_1)$ and there exist $(2, 1, pz_1)$ and $(2, \mu, pz_1)$ critical paths.*

Selection of a special color θ: Since $|F_{p'}(d_1)| \leq \Delta$, there exists a color θ missing at vertex p'. By $Lemma$ 5.45, $\theta \notin C - F_p(d_1) \subseteq F_{p'}(d_1)$. Thus $\theta \in F_p(d_1)$. Clearly $\theta \neq 2$ since $2 \in F_{p'}$ and $\theta \neq 1, \mu$ because $1, \mu \notin F_p(d_1)$ and hence by $Lemma$ 5.45 we have $1, \mu \in S_{pp'}(d_1)$. Further $\theta \neq \eta$. This is because by $Lemma$ 5.28, the $(\eta, 2)$ maximal bichromatic path starts at vertex q and contains the vertex y'_j. Clearly the first three vertices of this path are q, p, p'. Recall that the length of the segment of this path between vertices q and y'_j is at least three. Therefore $\eta \in S_{pp'}(d_1)$. Now without loss of generality let $d_1(p, z_2) = \theta(\neq 1, \eta, \mu, 2)$. Note that z_2 is a vertex different from z_1.

Note that with respect to the coloring $c_{1,\eta}$, the $(1, \eta, xy_1)$ critical path passes through the vertex z_1 (by (5.44)). This critical path cannot contain the vertex z_2. This is because if z_2 is an internal vertex of this critical path, then the edge pz_2 should be contained in the path, a contradiction since $c_{1,\eta}(p, z_2) = \theta \neq 1, \eta$. On the other hand if z_2 is an end vertex then it implies that $z_2 \in \{x, y_1\}$, a contradiction in view of $Lemma$ 5.29. Thus vertex z_2 is not contained in the $(1, \eta, xy_1)$ critical path. While getting the coloring d_1 from $c_{1,\eta}$, this path was broken due to the recoloring of $z_1z'_1$ and pz_1. It follows that the $(1, \eta)$ maximal bichromatic path that starts at vertex y_1 does not contain vertex z_2. Thus we can infer that,

Observation 5.47. *With respect to the coloring d_1, there cannot exist a $(1, \eta, y_1, z_2)$ maximal bichromatic path.*

Getting a valid coloring d_2 of $G_{1,\eta} - \{pz_2\}$ from d_1 of $G_{1,\eta} - \{pz_1\}$ by only local recolorings

We get a coloring d_1'' of $G_{1,\eta} - \{pz_1, pz_2\}$ from d_1 by discarding the edge pz_2. Note that the partial coloring d_1'' of $G_{1,\eta}$ is valid.

Now recolor the edge pz_1 with color the $special\ color$ θ to get a coloring d_2 of $G_{1,\eta} - \{pz_2\}$. Note that the color θ is a candidate for the edge pz_1 with respect to the coloring d_1'' since $d_1''(z_1, z_1') = 2$ and $\theta \notin F_p(d_1'')$ since we have removed the edge pz_2 (Recall that $d_1''(p, z_2) = \theta$). We claim that d_2 is valid also. Clearly if there is any bichromatic cycle created, then it has to be a $(\theta, 2)$ bichromatic cycle since $d_2(z_1, z_1') = 2$. Now this bichromatic cycle has to pass through vertex p' since $d_2(p, p') = 2$. But by the definition of color θ, it was not present at vertex p'. Thus there cannot be a $(\theta, 2)$ bichromatic cycle. It follows that the partial coloring d_2 of $G_{1,\eta} - \{pz_2\}$ is valid. Recall that by (5.6) that there exists no $(1, \eta, xy_1)$ critical path with respect to d_1. Note that to get d_2 from d_1, we just assigned $\theta(\neq 1, 2, \eta, \mu)$ to the edge pz_1 and removed the edge pz_2. Thus there is no chance of $(1, \eta, xy_1)$ critical path getting created with respect to d_2. Hence we have,

With respect to the partial coloring d_2, there does not exists any $(1, \eta, xy_1)$ critical path. (5.7)

Getting a valid coloring $c'_{1,\eta}$ of $G_{1,\eta}$ from d_2 of $G_{1,\eta} - \{pz_2\}$ by only local recolorings

Now we will show that we can give a valid color for the edge pz_2 to get a valid coloring for the graph $G_{1,\eta}$. We claim the following:

Lemma 5.48. *With respect to the coloring d_2 at least one of the colors 1, μ is valid for the edge pz_2. (Recall that by $Observation$ 5.46, $1, \mu \notin F_p(d_1)$ and therefore $1, \mu \notin F_p(d_2)$)*

Proof: Let $d_2(z_2, z_2') = \sigma$. Now if,

1. $\sigma = 2$. Recolor the edge pz_2 using color 1 to get a coloring d_3. The coloring d_3 is valid because if a bichromatic cycle gets formed it has to be $(1, 2)$ bichromatic cycle containing the vertex p implying that there was a $(2, 1, pz_2)$ critical path with respect to d_2. But by Observation 5.46, there was a $(2, 1, pz_1)$ critical path with respect to the coloring d_1 and hence with respect to the coloring d_2 (Note that to get d_2 from d_1, we

just assigned $d_1(p, z_2) = \theta$ ($\neq 1, 2, \eta, \mu$) to edge pz_1 and removed the edge pz_2. Thus the $(2, 1, pz_1)$ critical path is not altered during this recoloring). Thus in view of Fact 2.1, there cannot be any $(2, 1, pz_2)$ critical path with respect to d_2 since $z_1 \neq z_2$, a contradiction. Thus the coloring d_3 is valid.

2. $\sigma \in \{1, \mu\}$. Recolor the edge pz_2 using color $\{1, \mu\} - \{\sigma\}$ to get a coloring d_3. The coloring d_3 will be valid because if a bichromatic cycle gets formed it has to be $(1, \mu)$ bichromatic cycle containing the vertex p. But since color $\sigma \in \{1, \mu\}$ is not present at vertex p, such a bichromatic cycle is not possible.

3. $\sigma \notin \{1, 2, \mu\}$. Recolor the edge pz_2 using color 1 to get a coloring d'_2. If the coloring d'_2 is valid, then let $d_3 = d'_2$. Otherwise if the coloring d'_2 is not valid, then there has to be a $(\sigma, 1)$ bichromatic cycle. Now let $d'_2(p, z_j) = \sigma$. Then the bichromatic cycle passes through the vertex z_j and hence $d'_2(z_j, z'_j) = 1$, since $degree_G(z_j) = 2$. Now we recolor edge pz_2 with color μ to get a coloring d_3. If there is a bichromatic cycle formed with respect to the coloring d_3, then it has to be a (μ, σ) bichromatic cycle and hence it passes through the vertex z_j. But color μ is not present at z_j since $d'_2(z_j, z'_j) = 1$. Thus there cannot be any (μ, σ) bichromatic cycle. Hence the coloring d_3 is valid.

Thus either color 1 or μ is valid for the edge pz_2.

■

To get the coloring d_3 from d_2 we have only given a valid color for the edge pz_2 and have not altered the color of any other edge (i.e., only local recolorings are done). Recall that d_2 does not have any $(1, \eta, xy_1)$ critical path (by (5.7)). Note that $d_3(x, q) = 1$ and $d_3(q, p) = \eta$. If we give color $\mu \neq 1, \eta$ to the edge pz_2, there is no chance of a $(1, \eta, xy_1)$ critical path getting formed in d_3. On the other hand, by giving color 1 to the edge pz_2 if a $(1, \eta, xy_1)$ critical path gets formed, then it means that there exists a $(1, \eta, y_1, z_2)$ maximal bichromatic path with respect to d_2 and hence with respect to d_1. But by $Observation$ 5.47 such a bichromatic path does not exist. Now let $c'_{1,\eta} = d_3$. Thus we have,

With respect to the valid coloring $c'_{1,\eta}$ of $G_{1,\eta}$, there does not exists any $(1, \eta, xy_1)$ critical path. (5.8)

In getting $c'_{1,\eta}$ from $c_{1,\eta}$ we have done only local recolorings and thus by $Lemma$ 5.40 $c'_{1,\eta}$

can be derived from some valid coloring c'_1 of G_1. Note that we have not changed the color of the edge $y_1y'_1$ while getting $c'_{1,\eta}$ from $c_{1,\eta}$ since $y_1 \notin \{p, p', z_1, \ldots, z_k, z'_1, \ldots, z'_k\}$ (by $Lemma$ 5.29). Thus $c'_{1,\eta}(y_1, y'_1) = 1$. It follows that the $Critical\ Path\ Property$ of $c'_{1,\eta}$ is the same as $Critical\ Path\ Property$ of $c_{1,\eta}$. This implies that there exists a $(1, \eta, xy_1)$ critical path with respect to the coloring $c'_{1,\eta}$, a contradiction in view of (5.8).

This completes the proof. ■

5.3 Remark

Our result is tight since there are 2-degenerate graphs which require $\Delta + 1$ colors (e.g., cycle, non-regular subcubic graphs, etc.) our proof is constructive and yields an efficient polynomial time algorithm. It is easy to see that its complexity is $O(\Delta n^2)$. (We have presented the proof in a non-algorithmic way. But it is easy to extract the underlying algorithm from it.)

Chapter 6

Planar Graphs-General case

In this chapter we look at acyclic edge coloring of planar graphs.

6.1 Previous Results and Definitions

The acyclic chromatic index of planar graphs has been studied previously. Fiedorowicz, Hauszczak and Narayanan [24] gave an upper bound of $2\Delta + 29$ for planar graphs. Independently Hou, Wu, GuiZhen Liu and Bin Liu [29] gave an upper bound of $max(2\Delta - 2, \Delta + 22)$, which is the best known result up to now for planar graphs. Note that for $\Delta \geq 24$, it is equal to $2\Delta - 2$.

Now we give some definitions that are used in the proof.

Definition 6.1. ***Multisets and Multiset Operations:*** *A multiset is a* generalized *set where a member can appear multiple times in the set. If an element x appears t times in the multiset S, then we say that multiplicity of x in S is t. In notation $mult_S(x) = t$. The cardinality of a finite multiset S, denoted by $\parallel S \parallel$ is defined as $\parallel S \parallel = \sum_{x \in S} mult_S(x)$.Let S_1 and S_2 be two multisets. The reader may note that there are various possible ways to define union of S_1 and S_2. For the purpose of this paper we will define one such union notion- which we call as the $join$ of S_1 and S_2, denoted as $S_1 \uplus S_2$. The multiset $S_1 \uplus S_2$ will have all the members of S_1 as well as S_2. For a member $x \in S_1 \uplus S_2$, $mult_{S_1 \uplus S_2}(x) = mult_{S_1}(x) + mult_{S_2}(x)$. Clearly $\parallel S_1 \uplus S_2 \parallel = \parallel S_1 \parallel + \parallel S_2 \parallel$. We also need a specially defined notion of the multiset difference of S_1 and S_2, denoted by $S_1 \setminus S_2$. It is the multiset of elements of S_1 which are not in S_2, i.e., $x \in S_1 \setminus S_2$ iff $x \in S_1$ but $x \notin S_2$ and $mult_{S_1 \setminus S_2}(x) = mult_{S_1}(x)$.*

6.2 The Theorem

Theorem 6.2. *If G is a planar graph, then $a'(G) \leq \Delta + 12$.*

Proof: A well-known strategy that is used in proving coloring theorems in the context of planar graphs is to make use of induction combined with the fact that there are some *unavoidable* configurations in any planar graph. Typically the existence of these *unavoidable* configurations are proved using the so called *charging and discharging argument* (See [37], for a comprehensive exposition). Loosely speaking, for the purpose of this paper, a *configuration* is a set $\{v\} \cup N(v)$, where v is some vertex in G, along with some information regarding the degrees of the vertices in $\{v\} \cup N(v)$. For example, the following lemma illustrates how certain unavoidable configurations appear in a planar graph:

Lemma 6.3. *[40] Let G be a simple planar graph with $\delta \geq 2$, where δ is the minimum degree of graph G. Then there exists a vertex v in G with exactly $deg(v) = k$ neighbours $v_1, v_2, \ldots, v_k$ with $deg(v_1) \leq deg(v_2) \leq \ldots \leq deg(v_k)$ such that at least one of the following is true:*

(A1) $k = 2$,

(A2) $k = 3$ *and* $deg(v_1) \leq 11$,

(A3) $k = 4$ *and* $deg(v_1) \leq 7$, $deg(v_2) \leq 11$,

(A4) $k = 5$ *and* $deg(v_1) \leq 6$, $deg(v_2) \leq 7$, $deg(v_3) \leq 11$.

Let graph G be a minimum counter example with respect to the number of edges for the statement in Theorem 6.2. From $Lemma$ 6.3 we know that there exists a vertex v in G such that it belongs to one of the configurations $A1$-$A4$. We now delete the edge vv_1 to get a graph G', where v and v_1 are as in $Lemma$ 6.3. Since G was the minimum counter example, G' has an acyclic edge coloring using $\Delta(G') + 12$ colors. Let c' be such a coloring. Now if $\Delta(G') < \Delta(G)$, then we have at least one extra color for G and we can assign that color to edge vv_1 to get a valid coloring of G, a contradiction to the fact that G is a counter example. Thus we have $\Delta(G') = \Delta(G) = \Delta$. To prove the theorem for G, we may assume that G is 2-$connected$ since if there are cut vertices in G, the acyclic edge coloring of the blocks $B_1, B_2 \ldots B_k$ of G can easily be extended to G. Thus we have, $\delta(G) \geq 2$. We present the proof in two parts based on which configuration the vertex v belongs to - The first part deals with the case when there exists a vertex v that belongs to configuration $A2$,$A3$ or $A4$ and the second part deals with the case when there does not exists any vertex v in G that belongs to configuration $A2$,$A3$ or $A4$.

6.2.1 There exists a vertex v that belongs to configuration $A2$,$A3$ or $A4$

Claim 6.4. *For any valid coloring c' of G', $|F_v \cap F_{v_1}| \geq 2$.*
Proof: Suppose not. The case $|F_v \cap F_{v_1}| = 0$ is trivial. The reader can verify from close examination of configurations $A2$-$A4$ that $|F_v \cup F_{v_1}|$ will be maximum for configuration $A2$ and therefore $|F_v \cup F_{v_1}| = |F_v| + |F_{v_1}| \leq 2 + 10 = 12$. Thus there are Δ candidate colors for the edge vv_1 and by $Lemma$ 2.3 all the candidate colors are valid, a contradiction to the assumption that G is a counter example. Thus we have $|F_v \cap F_{v_1}| = 1$. In this case it is easy to see that $|F_v \cup F_{v_1}| = |F_v| + |F_{v_1}| - |F_v \cap F_{v_1}| \leq 11$ and hence there are at least $\Delta + 1$ candidate colors for the edge vv_1. Let $F_v \cap F_{v_1} = \{\alpha\}$ and let $u \in N(v)$ be a vertex such that $c'(v, u) = \alpha$. Now if none of the $\Delta+1$ candidate colors are valid for the edge vv_1, then by $Fact$ 2.5, for each $\gamma \in C - (F_v \cup F_{v_1})$, there exists a (α, γ, vv_1) critical path. Since $c'(v, u) = \alpha$, we have all the critical paths passing through the vertex u and hence $S_{vu} \subseteq C - (F_v \cup F_{v_1})$. This implies that $|S_{vu}| \geq |C - (F_v \cup F_{v_1})| \geq (\Delta + 12) - 11 = \Delta + 1$, a contradiction since $|S_{vu}| \leq \Delta - 1$. Thus we have a valid color for the edge vv_1, a contradiction to the assumption that G is a counter example. Thus $|F_v \cap F_{v_1}| \geq 2$. □

Let S_v be a multiset defined as $S_v = S_{vv_2} \uplus S_{vv_3} \uplus \ldots \uplus S_{vv_k}$. In view of $Claim$ 6.4 and Lemma 6.3, $2 \leq |F_v \cap F_{v_1}| \leq 4$. We consider each case separately.

case 1: $|F_v \cap F_{v_1}| = 2$

Let $F_v \cap F_{v_1} = \{1, 2\}$ and let $v', v'' \in N_{G'}(v)$ and $u', u'' \in N_{G'}(v_1)$ be such that $c'(v, v') = c'(v_1, u') = 1$ and $c'(v, v'') = c'(v_1, u'') = 2$. It is easy to see that $|F_v \cup F_{v_1}| \leq 10$. Thus there are at least $\Delta + 2$ candidate colors for the edge vv_1. If any of the candidate colors are valid for the edge vv_1, we are done. Thus none of the candidate colors are valid for the edge vv_1. This implies that there exist either a $(1, \theta, vv_1)$ or $(2, \theta, vv_1)$ critical path for each candidate color θ.

Claim 6.5. *With respect to the coloring c', the multiset S_v contains at least $|F_{v_1}| - 1$ colors from F_{v_1}.*
Proof: Suppose not. Then there are at least two colors in F_{v_1} which are not in S_v. Let ν and μ

be any two such colors. Now assign colors ν and μ to the edges vv' and vv'' respectively to get a coloring c''. Now since $\nu, \mu \notin S_v$, we have $\nu \notin S_{vv'}$ and $\mu \notin S_{vv''}$. Moreover $\mu, \nu \notin \{1, 2\}$. Thus the recoloring c'' is proper. Now we claim that the coloring c'' is acyclic also. Suppose not. Then there has to be a bichromatic cycle containing at least one of the colors ν and μ. Clearly this cannot be a (ν, μ) bichromatic cycle since $\mu \notin S_{vv'}$. Therefore it has to be a (ν, λ) or (μ, λ) bichromatic cycle where $\lambda \in F_v(c'') - \{\nu, \mu\}$. Let u be a vertex such that $c''(v, u) = \lambda$. This means that there was already a (λ, ν, vv') or (λ, μ, vv'') critical path with respect to the coloring c'. This implies that $\nu \in S_{vu}$ or $\mu \in S_{vu}$, implying that $\nu \in S_v$ or $\mu \in S_v$, a contradiction. Thus the coloring c'' is acyclic. Let $u_1, u_2 \in N_{G'}(v_1)$ be such that $c''(v_1, u_1) = \nu$ and $c''(v_1, u_2) = \mu$.

Note that $|F_v \cup F_{v_1}| \leq 10$ (The maximum value of $|F_v \cup F_{v_1}|$ is attained when the graph has configuration $A2$). Therefore there are at least $\Delta + 2$ candidate colors for the edge vv_1. If any of the candidate colors are valid for the edge vv_1, then we are done as this is a contradiction to the assumption that G is a counter example. Thus none of the candidate colors are valid for the edge vv_1 and therefore there exist either a (ν, θ, vv_1) or (μ, θ, vv_1) critical path for each candidate color θ. Let C_ν and C_μ respectively be the set of candidate colors which are forming critical paths with colors ν and μ. Then clearly $C_\nu \subseteq S_{v_1u_1}$ and $C_\mu \subseteq S_{v_1u_2}$ since $c''(v_1, u_1) = \nu$ and $c''(v_1, u_2) = \mu$. Now we *exchange the colors* of the edges vv' and vv'' to get a modified coloring c. Note that c is proper since $\mu \notin S_{vv'}$ and $\nu \notin S_{vv''}$. By Lemma 2.8, all (ν, β, vv_1) critical paths where $\beta \in C_\nu$ and all (μ, γ, vv_1) critical paths where $\gamma \in C_\mu$ are broken. Now if none of the colors in C_ν are valid for edge vv_1, then it means that for each $\beta \in C_\nu$, there exists a (μ, β, vv_1) critical path with respect to coloring c, implying that $C_\nu \subseteq S_{v_1u_2}$. Since the recoloring involved no candidate colors, we still have $C_\mu \subseteq S_{v_1u_2}$. Thus we have $(C_\nu \cup C_\mu) \subseteq S_{v_1u_2}$. But $|C_\nu \cup C_\mu| \geq \Delta + 2$ which implies that $|S_{v_1u_2}| \geq \Delta + 2$, a contradiction since $|S_{v_1u_2}| \leq \Delta - 1$. □

Claim 6.6. *With respect to the coloring c', there exists at least two colors α and β in $C - F_{v_1}$ with multiplicity at most one in S_v.*

Proof: In view of Claim 6.5 we have $\sum_{x \in C - F_v} mult_{S_v}(x) = \parallel S_v \parallel - (|F_v| - 1)$. Thus if $\parallel S_v \parallel - (|F_{v_1}| - 1) \leq 2|(C - F_{v_1})| - 3$, then there exist at least two colors α and β in $C - F_{v_1}$ with multiplicity at most one in S_v. Thus it is enough to prove $\parallel S_v \parallel \leq 2|C| - |F_{v_1}| - 4 \leq 2\Delta + 24 - |F_{v_1} - 4 = 2\Delta + 20 - |F_{v_1}|$. Now we can easily verify that $\parallel S_v \parallel + |F_{v_1}| \leq 2\Delta + 20$ for configurations $A2 - A4$ as follows:

- For $A2$, $\parallel S_v \parallel + |F_{v_1}| \leq (deg(v_2)-1)+(deg(v_3)-1)+|F_{v_1}| = (\Delta-1)+(\Delta-1) \ +10 = 2\Delta + 8$.

- For $A3$, $\parallel S_v \parallel + |F_{v_1}| \leq (deg(v_2) - 1) + (deg(v_3) - 1) + (deg(v_4) - 1) + |F_{v_1}| = 10 + (\Delta - 1) + (\Delta - 1) \ \ + 6 = 2\Delta + 14$.

- For $A4$, $\parallel S_v \parallel + |F_{v_1}| \leq (deg(v_2) - 1) + (deg(v_3) - 1) + (deg(v_4) - 1) + (deg(v_5) - 1) + |F_{v_1}| = 6 + 10 + (\Delta - 1) + (\Delta - 1) \ \ + \ \ 5 = 2\Delta + 19$.

□

The colors α and β of $Claim$ 6.6 are crucial to the proof. Now we make another claim regarding α and β:

Claim 6.7. *With respect to the coloring c', α and $\beta \in F_v$.*

Proof: Without loss of generality, let $\alpha \notin F_v$. Then recalling that $\alpha \notin F_{v_1}$, α is a candidate for the edge vv_1. If it is not valid, then there exists either a $(1, \alpha, vv_1)$ or $(2, \alpha, vv_1)$ critical path with respect to c'. Since the multiplicity of α in S_v is at most one, we have the color α in exactly one of $S_{vv'}$ or $S_{vv''}$. Without loss of generality let $\alpha \in S_{vv''}$. Hence there exists either a $(2, \alpha, vv_1)$ critical path with respect to c'.

Now recolor the edge vv' with color α to get a coloring c. It is obvious that the recoloring c is proper since $\alpha \notin F_v(c')$ and $\alpha \notin S_{vv'}(c')$. It is also valid since if a bichromatic cycle gets formed due to this recoloring, it has to be a (α, γ) bichromatic cycle for some $\gamma \in F_v(c) - c(v, v')$. Let $a \in N_{G'}(v)$ be such that $c(v, a) = \gamma$. Then the (α, γ) bichromatic cycle should contain the edge va and therefore $\gamma \in S_{va}$ with respect to c. But we know that v'' is the only vertex in $N_{G'}(v)$ such that $\alpha \in S_{vv''}$. Therefore $a = v''$. This implies that $\gamma = 2$ and there existed a $(2, \alpha, vv')$ critical path with respect to the coloring c'. This is a contradiction to the Fact 2.1 since there already existed a $(2, \alpha, vv_1)$ critical path with respect to the coloring c'. Thus the recoloring c is valid. Now with respect to the coloring c, $|F_v \cap F_{v_1}| = 1$, a contradiction to $Claim$ 6.4. □

Note that $\alpha, \beta \notin \{1, 2\}$ since $\alpha, \beta \notin F_{v_1}$. In view of $Claim$ 6.7, we have $\{1, 2, \alpha, \beta\} \subseteq F_v$ and thus $|F_v| \geq 4$, which implies that $deg(v) \geq 5$. Thus the vertex v belongs to configuration $A4$. Therefore $deg(v) = 5$ and $F_v = \{1, 2, \alpha, \beta\}$. There are at least $\Delta + 12 - (5 + 4 - 2) = \Delta + 5$

candidate colors for the edge vv_1. Also recall that $deg(v_2) \leq 7$, $c'(v,v') = c'(v_1,u') = 1$ and $c'(v,v'') = c'(v_1,u'') = 2$.

Claim 6.8. *With respect to the coloring* c', $v_2 \notin \{v', v''\}$.

Proof: Suppose not. Then without loss of generality let $v_2 = v'$ and $c'(v,v_2) = 1$. Now if none of the $\Delta + 5$ candidate colors are valid for the edge vv_1, then they all form critical paths that contain either the edge vv' or the edge vv''. Now $|S_{vv'}|$ + $|S_{vv''}| \leq 6 + \Delta - 1 = \Delta + 5$. Since each of the $\Delta + 5$ candidate colors has to be present in either in $S_{vv'}$ or $S_{vv''}$, we infer that $S_{vv''} \cup S_{vv'}$ is exactly the set of candidate colors, i.e., $|S_{vv'}|$ + $|S_{vv''}|$ = $\Delta + 5$. This requires that $|S_{vv'}| = 6$, $|S_{vv''}| = \Delta - 1$ and $S_{vv''} \cap S_{vv'} = \emptyset$. Since for each $\gamma \in S_{vv''}$, we have $(2, \gamma, vv_1)$ critical path containing u'', we can infer that $S_{vv''} \subseteq S_{v_1u''}$ (Recall that $c'(v_1,u'') = 2$). But since $|S_{v_1u''}| \leq \Delta - 1$, we infer $S_{vv''} = S_{v_1u''}$. Thus we have $S_{v_1u''} \cap S_{vv'} = S_{vv''} \cap S_{vv'} = \emptyset$.

Now we exchange the colors of the edges vv' and vv'' to get a coloring c i.e., $c(v,v') = 2$ and $c(v,v'') = 1$. The coloring c is proper since $2 \notin S_{vv'}(c')$ and $1 \notin S_{vv''}(c')$ (Recall that $S_{vv'}(c')$ and $S_{vv''}(c')$ contain only candidate colors). The coloring is also valid: If a bichromatic cycle gets formed it has to be a $(1, \eta)$ or $(2, \eta)$ bichromatic cycle where $\eta \in F_v$. Clearly it cannot be a $(1, 2)$ bichromatic cycle since $1 \notin S_{vv'}(c)$ and therefore $\eta = \alpha$ or β (Recall that $F_v = \{1, 2, \alpha, \beta\}$). This implies that either α or β belongs to $S_{vv'} \cup S_{vv''}$. But we know that $S_{vv'} \cup S_{vv''}$ is exactly the set of candidate colors for the edge vv_1, a contradiction since $\alpha, \beta \in F_v$ cannot be candidate colors for the edge vv_1. Therefore the coloring c is acyclic. By $Lemma$ 2.8, all the existing critical paths are broken. Now consider a color $\gamma \in S_{vv'}$. If it is still not valid then there has to be a $(2, \gamma, vv_1)$ critical path since $c(v,v') = 2$ and $\gamma \notin S_{vv''}(c)$. This implies that $\gamma \in S_{v_1u''}(c)$, a contradiction since $S_{v_1u''}(c) \cap S_{vv'}(c) = \emptyset$. Thus we have a valid color for the edge vv_1, a contradiction to the assumption that G is a counter example. Thus $v_2 \notin \{v', v''\}$. □

From $Claim$ 6.8, we infer that $c'(v,v_2) \notin F_v \cap F_{v_1}$ since $F_v \cap F_{v_1} = \{c'(v,v'), c(v,v'')\} = \{1, 2\}$. Therefore we have $c(v,v_2) \in \{\alpha, \beta\}$ since $F_v = \{1, 2, \alpha, \beta\}$. Without loss of generality let $c(v,v_2) = \alpha$. We know that the color β can be in at most one of $S_{vv'}$ and $S_{vv''}$ by Claim 6.6. Now let v' be such that $\beta \notin S_{vv'}$. Note that $C - (S_{vv'} \cup F_v \cup F_{v_1}) \neq \emptyset$ since $|S_{vv'} \cup F_v \cup F_{v_1}| \leq \Delta - 1 + 4 + 5 - 2 = \Delta + 6$. Assign a color $\theta \in C - (S_{vv'} \cup F_v \cup F_{v_1})$ to the edge vv' to get a coloring c''. If it is valid, then let $c = c''$.

If the recoloring is not valid then there has to be a bichromatic cycle created due to the recoloring. Now the bichromatic cycle should involve one of the colors 2, α, β along with θ. Since the bichromatic cycle contains a color from $S_{vv'}$ and $\beta \notin S_{vv'}$, it cannot be a (θ, β) bichromatic cycle. Now with respect to the coloring c', color θ was not valid for the edge vv_1 implying that there existed either a $(1, \theta, vv_1)$ or a $(2, \theta, vv_1)$ critical path. But $(1, \theta, vv_1)$ critical path was not possible since $\theta \notin S_{vv'}$ by the choice of θ. Thus there existed a $(2, \theta, vv_1)$ critical path with respect to c'. Thus by Fact 2.1, there cannot be a $(2, \theta, vv')$ critical path with respect to c' and hence there cannot be a $(2, \theta)$ bichromatic cycle in c'' formed due to the recoloring. Thus if there is a bichromatic cycle formed, then it has to be a (α, θ) bichromatic cycle, which implies that $\alpha \in S_{vv'}$.

Now taking into account the fact that α is in $S_{vv'}$ as well as F_v, we get $|S_{vv'} \cup F_v \cup F_{v_1}| \leq \Delta - 1 + 4 + 5 - 2 - 1 = \Delta + 5$ and therefore $|S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2}| \leq \Delta + 5 + 6 = \Delta + 11$. Thus $C - (S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2}) \neq \emptyset$. Now recolor the edge vv' using a color $\gamma \in C - (S_{vv'} \cup F_v \cup F_{v_1} \cup S_{vv_2})$ to get a coloring c. Clearly the recoloring is proper. It is also valid since if a bichromatic cycle gets formed it has to be a (α, γ) bichromatic cycle (Note that the $(2, \gamma)$ and (β, γ) bichromatic cycles are argued out as before). But $\gamma \notin S_{vv_2}$, a contradiction. Thus the coloring c is acyclic.

With respect to the coloring c we have $|F_v \cap F_{v_1}| = 1$, a contradiction to Claim 6.4.

case 2: $|F_v \cap F_{v_1}| = 3$

Note that in this case $|F_v| \geq 3$ and therefore $deg(v) \geq 4$. Thus v belongs to either configuration $A3$ or $A4$. Let S'_v be a multiset defined by $S'_v = S_v \setminus (F_{v_1} \cup F_v)$. Let $v', v'', v''' \in N_{G'}(v)$ be such that $\{c(v, v'), c(v, v''), c(v, v''')\} = F_v \cap F_{v_1}$. Also let $c(v, v') = 1$, $c(v, v'') = 2$ and $c(v, v''') = 3$.

Claim 6.9. *With respect to c', $\| S'_v \| \leq 2\Delta + 11$.*

Proof: When $deg(v) = 4$, it is clear that $\| S'_v \| \leq (deg(v_2)-1)+(deg(v_3)-1)+(deg(v_4)-1) \leq 10 + \Delta - 1 + \Delta - 1 = 2\Delta + 8$. On the other hand when $deg(v) = 5$, try to recolor one of the edges vv', vv'', vv''' using a color in $C - (F_v \cup F_{v_1})$. There are $\Delta + 6$ colors in $C - (F_v \cup F_{v_1})$ and if any of these colors is valid for one of vv', vv'' or vv''', then the situation reduces to case 1 i.e., $|F_v \cap F_{v_1}| = 2$. Otherwise there has to be a bichromatic cycle formed during each recoloring. Since such a bichromatic cycle has to be (γ_1, γ_2) bichromatic cycle where γ_1 is the color used in the recoloring and $\gamma_2 \in F_v - \{\gamma_1\}$, we infer that $S_{vv'}$, $S_{vv''}$ and $S_{vv'''}$ contain at least one color from F_v. Thus we have $\| S'_v \| \leq \| S_v \| -3 \leq (deg(v_2) - 1) + (deg(v_3) - 1) +$

$(deg(v_4)-1)+(deg(v_5)-1)-3 \leq 6+10+\Delta-1+\Delta-1-3 = 2\Delta+11$. □

Claim 6.10. *With respect to c', there exists at least one color $\alpha \in C-(F_v \cup F_{v_1})$ with multiplicity at most one in S'_v.*

Proof: Since v belongs to either configuration $A3$ or configuration $A4$, we have $|F_v \cup F_{v_1}| \leq 9-3=6$. Thus $|C-(F_v \cup F_{v_1})| \leq \Delta+6$. By $Claim$ 6.9 we have $\| S'_v \| \leq 2\Delta+11$ and from this it is easy to see that there exists at least one color $\alpha \in C-(F_v \cup F_{v_1})$ with multiplicity at most one in S'_v. □

Note that $\alpha \in C-(F_v \cup F_{v_1})$, where α is the color from $Claim$ 6.10 is a candidate color for the edge vv_1. If it is not valid then there has to be a (θ, α, vv_1) critical path, where $\theta \in \{1,2,3\}$. By $Claim$ 6.10, α can be present in at most one of $S_{vv'}$, $S_{vv''}$ and $S_{vv'''}$. Without loss of generality let $\alpha \in S_{vv''}$. Thus there exists a $(2, \alpha, vv_1)$ critical path with respect to the coloring c'. Recolor the edge vv' using the color α to get a coloring c. Clearly the recoloring is proper since $\alpha \notin S_{vv'}$ and $\alpha \notin F_v$. The recoloring is valid since if a bichromatic cycle gets formed then it has to contain the color α as well as a color $\gamma \in F_v(c)-\{\alpha\}$. If $\gamma = c(v,w)$, then $\alpha \in S_{vw}$, for the (α, γ) bichromatic cycle to get formed. But v'' is the only vertex in $N_{G'}(v)$ such that $\alpha \in S_{vv''}$. Thus $w = v''$, $\gamma = 2$ and it has to be a $(\alpha, 2)$ bichromatic cycle. This means that there existed a $(2, \alpha, vv')$ critical path with respect to the coloring c', a contradiction by $Fact$ 2.1 since there already existed a $(2, \alpha, vv_1)$ critical path with respect to the coloring c'. Thus the coloring c is acyclic. This reduces the situation to case 1.

case 3: $|F_v \cap F_{v_1}| = 4$

Note that in this case $|F_v| \geq 4$ and since $deg(v) \leq 5$, we have $deg(v) = 5$. In other words, v belongs to configuration $A4$. Let S'_v be a multiset defined by $S'_v = S_v \setminus (F_{v_1} \cup F_v)$. Also let $c(v,v_2)=1$, $c(v,v_3)=2$, $c(v,v_4)=3$ and $c(v,v_5)=4$.

Now try to recolor an edge incident on v with a candidate color from $C-(F_v \cup F_{v_1})$. If the recoloring is valid then the situation reduces to case 2. Otherwise there has to be a bichromatic cycle created due to recoloring with one of the colors from F_v. This implies that $F_v \cap S'_v \neq \emptyset$. Thus we have $\| S'_v \| \leq \| S_v \| -1 \leq (deg(v_2)-1)+(deg(v_3)-1)+(deg(v_4)-1)+(deg(v_5)-1) \leq 6+10+\Delta-1+\Delta-1-1 = 2\Delta+13$. Now since there are $|C-(F_v \cup F_{v_1})| \geq \Delta+12-(4+5-4) = \Delta+7$ candidate colors and $\| S'_v \| \leq 2\Delta+13$, it is easy to see that there exists at least one candidate color α with multiplicity at most one in S'_v.

Note that $\alpha \in C - (F_v \cup F_{v_1})$ is a candidate color for the edge vv_1. If it is not valid then there has to be a (θ, α, vv_1) critical path, where $\theta \in \{1, 2, 3, 4\}$. We know that α can be present in at most one of S_{vv_2}, S_{vv_3}, S_{vv_4} and S_{vv_5}. Without loss of generality let $\alpha \in S_{vv_3}$. Thus there exists a $(2, \alpha, vv_1)$ critical path with respect to the coloring c'. Recolor the edge vv_2 using the color α to get a coloring c. Clearly the recoloring is proper since $\alpha \notin S_{vv_2}$ and $\alpha \notin F_v$. The recoloring is valid since if a bichromatic cycle gets formed then it has to contain the color α as well as a color $\gamma \in F_v(c) - \{\alpha\}$. If $\gamma = c(v, w)$, then $\alpha \in S_{vw}$, for the (α, γ) bichromatic cycle to get formed. But v_3 is the only vertex in $N_{G'}(v)$ such that $\alpha \in S_{vv_3}$. Thus $w = v_3$, $\gamma = 2$ and it has to be a $(\alpha, 2)$ bichromatic cycle. This means that there existed a $(2, \alpha, vv_2)$ critical path with respect to the coloring c', a contradiction by $Fact$ 2.1 since there already existed a $(2, \alpha, vv_1)$ critical path with respect to the coloring c'. Thus the coloring c is acyclic. This reduces the situation to case 2.

6.2.2 There exists no vertex v that belongs to configuration $A2$,$A3$ or $A4$

Then clearly by $Lemma$ 6.3, we can assume that there is a vertex v that belongs to configuration $A1$, i.e., $deg(v) = 2$. Now delete all the degree 2 vertices from G to get a graph H. Now since the graph H is also planar, there exists a vertex v' in H such that v' belongs to one of the configurations $A1 - A4$, say A'. The vertex v' was not already in configuration A' in G. This means that the degree of at least one of the vertices of the configuration A' i.e., $\{v'\} \cup N_H(v')$, got decreased by the removal of 2-degree vertices. Let $P = \{x \in \{v'\} \cup N_H(v') : d_H(x) < d_G(x)\}$. Let u be the minimum degree vertex in P in the graph H. Now it is easy to see that $d_H(u) \leq 11$ since v' did not belong to A' in G.

Let $N'(u) = \{x | x \in N_G(u)$ and $d_G(u) = 2\}$. Let $N''(u) = N_G(u) - N'(u)$. It is obvious that $N''(u) = N_H(u)$.

Since $u \in P$ and $d_H(u) \leq 11$, we have $|N'(u)| \geq 1$ and $N''(u) \leq 11$. In G let $u' \in N'(u)$ be a two degree neighbour of u such that $N(u') = \{u, u''\}$. Now by induction $G - \{uu'\}$ is acyclically edge colorable using $\Delta + 12$ colors. Let c' be such a coloring. With respect to a partial coloring c' let $F'_u(c') = \{c'(u, x) | x \in N'(u)\}$ and $F''_u(c') = \{c'(u, x) | x \in N''(u)\}$. Now if $c(u', u'') \notin F_u$ we are done since $|F_u \cup F_{u'}| \leq \Delta$ and thus there are at least 12 candidate colors which are also valid by $Lemma$ 2.3.

We know that $|F''_v| \leq 11$. If $c'(u', u'') \in F'_v$, then let $c = c'$. Else if $c'(u', u'') \in F''_v$, then recolor edge $u'u''$ using a color from $C - (S_{u'u''} \cup F''_v)$ to get a coloring c (Note that

$|C-(S_{u'u''}\cup F''_v)| \geq \Delta+12-(\Delta-1+11) = 2$ and since u' is a pendant vertex in $G-\{uu'\}$ the recoloring is valid). Now if $c(u',u'')\notin F_u$ the proof is already discussed. Thus $c(u',u'')\in F'_u$.

With respect to coloring c, let $a\in N'(v)$ be such that $c(v,a)=c(u',u'')=1$. Now if none of the candidate colors in $C-(F_u\cup F_{u'})$ are valid for the edge uu', then by $Fact$ 2.5, for each $\gamma\in C-(F_u\cup F_{u'})$, there exists a $(1,\gamma,uu')$ critical path. Since $c'(v,a)=1$, we have all the critical paths passing through the vertex a and hence $S_{va}\subseteq C-(F_u\cup F_{u'})$. This implies that $|S_{va}|\geq|C-(F_u\cup F_{u'})|\geq\Delta+12-(1+\Delta-1-1)=13$, a contradiction since $|S_{va}|=1$. Thus we have a valid color for the edge uu', a contradiction to the assumption that G is a counter example.

■

Chapter 7

Triangle Free Planar Graphs

In this chapter we look at acyclic edge coloring of triangle free planar graphs.

7.1 Previous Results

The acyclic chromatic index of special classes of planar graphs characterized by some lower bounds on girth or the absence of short cycles have also been studied. In [29] an upper bound of $\Delta + 2$ for planar graphs of girth at least 5 has been proved. Fiedorowicz and Borowiecki [23] proved an upper bound of $\Delta + 1$ for planar graphs of girth at least 6 and an upper bound of $\Delta + 15$ for planar graphs without cycles of length 4. In [24], an upper bound of $\Delta + 6$ for triangle free planar graphs has been proved. In this chapter we improve the bound to $\Delta + 3$. In fact we prove a more general theorem as described below:

Definition 7.1. $Property\ A$: *Let G be a simple graph. If every induced subgraph H of G satisfies the condition $|E(H)| \leq 2|V(H)| - 1$, we say that the graph G satisfies $Property\ A$. If G satisfies $Property\ A$, then every subgraph of G also satisfies $Property\ A$.*

Note that triangle free planar graphs, 2-degenerate graphs, 2-fold graphs (union of two forests), etc. are some classes of graphs which satisfy $Property\ A$. The earlier known bound for these classes of graphs was $\Delta + 6$ by [24].

The following is the main result of [11]. We will need this result for proving our theorem.

Lemma 7.2. *[11] Let G be a connected graph on n vertices, $m \leq 2n - 1$ edges and maximum degree $\Delta \leq 4$, then $a'(G) \leq 6$.*

7.2 The Theorem

Theorem 7.3. *If a graph G satisfies $Property\ A$, then $a'(G) \leq \Delta(G) + 3$.*

Proof: A well-known strategy that is used in proving coloring theorems in the context of sparse graphs is to make use of induction combined with the fact that there are some *unavoidable* configurations in any such graphs. Typically the existence of these *unavoidable* configurations are proved using the so called *charging and discharging argument* (See [37], for a comprehensive exposition). $Lemma$ 7.4 will establish that one of the five configurations $B1, \ldots, B5$ is unavoidable in any graph G that satisfies $Property\ A$. Loosely speaking, for the purpose of this paper, a *configuration* is a subset Q of V, where one special vertex $v \in Q$ is called the $pivot$ of the configuration and $Q = \{v\} \cup N(v)$. Besides v, one more vertex in Q will be given a special status: This vertex, called the *co-pivot* of the configuration, is selected such that it is a vertex of smallest degree in $N(v)$ and will be denoted by u. Moreover the vertices of $N(v)$ will be partitioned into two sets namely $N'(v)$ and $N''(v)$. The members of $N'(v)$ and $N''(v)$ are explicitly defined for each configuration.

Lemma 7.4. *Let G be a simple graph such that $|E(G)| \leq 2|V(G)| - 1$ with minimum degree $\delta \geq 2$. Then there exists a vertex v in G with $k = deg(v)$ neighbours such that at least one of the following is true:*

(B1) $k = 2$,

(B2) $k = 3$ *with* $N(v) = \{u, v_1, a\}$ *such that* $deg(u)$, $deg(v_1) \leq 4$. $N'(v) = \{u, v_1\}$ *and* $N''(v) = \{a\}$,

(B3) $k = 5$ *with* $N(v) = \{u, v_1, v_2, a, b\}$ *such that* $deg(u)$, $deg(v_1)$, $deg(v_2) \leq 3$. $N'(v) = \{u, v_1, v_2\}$ *and* $N''(v) = \{a, b\}$,

(B4) $k = 6$ *with* $N(v) = \{u, v_1, v_2, v_3, v_4, a\}$ *such that* $deg(u)$, $deg(v_1)$, $deg(v_2)$, $deg(v_3)$, $deg(v_4) \leq 3$. $N'(v) = \{u, v_1, v_2, v_3, v_4\}$ *and* $N''(v) = \{a\}$,

(B5) $k \geq 7$ *with* $N(v) = \{u, v_1, v_2, \ldots, v_{k-1}\}$ *such that* $deg(u)$, $deg(v_1)$, $deg(v_2)$, $deg(v_3)$, $\ldots$, $deg(v_{k-1}) \leq 3$. $N'(v) = \{u, v_1, v_2, \ldots, v_{k-1}\}$.

Proof: We use the discharging method to prove the lemma. Let $G = (V, E)$, $\delta \geq 2$, $|V| = n$ and $|E| = m \leq 2n - 1$. We define a mapping $\phi : V \longmapsto \mathbb{R}$ using the rule $\phi(v) = deg(v) - 4$ for each $v \in V$. The value $\phi(v)$ is called the charge on the vertex v. Since $m \leq 2n - 1$, it is easy to see that $\sum_{v \in V} \phi(v) \leq -2$. Now we redistribute the charges on the vertices using the following rule. (This procedure is usually known as *discharging*: Note that the total charge has to remain same after the discharging.)

- If vertex v has degree at least 5, then it gives a charge of $\frac{1}{2}$ to each of its 3-degree neighbours.

After $discharging$, each vertex v has a new charge $\phi'(v)$. Now since the total charge is conserved, we have $\sum_{v \in V} \phi(v) = \sum_{v \in V} \phi'(v) \leq -2$. Now suppose the graph G has none of the configurations $B1, \ldots, B5$. Then we will show that for each vertex v of G, $\phi'(v) \geq 0$ and therefore $\sum_{v \in V} \phi'(v) \geq 0$, a contradiction. Since G does not have configuration $B1$, we have $\delta \geq 3$. Now we calculate the charge on each vertex v of G as follows:

- If $deg(v) = 3$: Since G does not have configuration $B2$, at least two of the neighbours have degree at least 5. Thus v receives a charge of $\frac{1}{2}$ each from at least two of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 + 2 \cdot \frac{1}{2} = 0$.
- If $deg(v) = 4$: A four degree vertex does not give or receive any charge. Thus $\phi'(v) = \phi(v) = deg(v) - 4 = 0$.
- If $deg(v) = 5$: Since G does not have configuration B3, at most two of the neighbours have degree 3. Thus v gives a charge of $\frac{1}{2}$ each to at most two of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 - 2 \cdot \frac{1}{2} = 0$.
- If $deg(v) = 6$: Since G does not have configuration B4, at most four of the neighbours have degree 3. Thus v gives a charge of $\frac{1}{2}$ each to at most four of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 - 4 \cdot \frac{1}{2} = 0$.
- If $deg(v) \geq 7$: Since G does not have configuration B5, at most $deg(v) - 1$ of the neighbours have degree 3. Thus v gives a charge of $\frac{1}{2}$ each to at most $deg(v) - 1$ of its neighbours. Thus $\phi'(v) \geq deg(v) - 4 - (deg(v) - 1) \cdot \frac{1}{2} = \frac{1}{2}(deg(v) - 7) \geq 0$.

Thus we have established that $\phi'(v) \geq 0, \forall v \in V$ and therefore $\sum_{v \in V} \phi'(v) \geq 0$, a contradiction.

■

We prove the theorem by way of contradiction. Let G be a minimum counter example (with respect to the number of edges) for the theorem statement among the graphs satisfying $Property\ A$. Clearly G is 2-$connected$ since if there are cut vertices in G, the acyclic edge coloring of the blocks $G_1, G_2, \ldots, G_k$ of G can easily be extended to G (Note that each block satisfies the $Property\ A$ since they are subgraphs of G). Thus we have, $\delta(G) \geq 2$. Also from $Lemma$ 7.2, we know that $a'(G) \leq \Delta+3$, when $\Delta \leq 4$. Therefore we can assume that $\Delta \geq 5$. Thus we have,

Assumption 7.5. *For the minimum counter example* G*,* $\delta(G) \geq 2$ *and* $\Delta(G) \geq 5$*.*

By $Lemma$ 7.4, graph G has a vertex v, such that it is the pivot of one of the configurations $B1, \ldots, B5$. We present the proof in two parts based on the configuration that v belongs to. The first part deals with the case when G has a vertex v that belongs to configuration $B2$, $B3$, $B4$ or $B5$ and the second part deals with the case when G does not have a vertex v that belongs to configuration $B2$, $B3$, $B4$ or $B5$.

7.2.1 There exists a vertex v that belongs to configuration $B2$, $B3$, $B4$ or $B5$

Let v be a vertex such that it is the pivot of one of the configurations $B2, \ldots, B5$ and let u be the co-pivot. Since G is a minimum counter example, the graph $G - \{vu\}$ is acyclically edge colorable using $\Delta+3$ colors. Let c' be a valid coloring of $G-\{vu\}$ and hence a partial coloring of G. We now try to extend c' to a valid coloring of G. With respect to the partial coloring c' let $F'_v(c') = \{c'(v,x) | x \in N'(v)\}$ and $F''_v(c') = \{c'(v,x) | x \in N''(v)\}$ i.e., $F''_v = F_v - F'_v$.

Claim 7.6. *With respect to any valid coloring* c' *of* $G - \{uv\}$*,* $|F_u \cap F_v| \geq 1$

Proof: Suppose not. Then $S_{vu} \cap S_{uv} = \emptyset$ and by $Lemma$ 2.3, all the candidate colors are valid for the edge vu. It is easy to verify that irrespective of which configuration v belongs to, $|F_u \cup F_v| \leq \Delta - 1 + 2 = \Delta + 1$. Therefore there are at least two candidate colors for the edge vu which are also valid, a contradiction to the assumption that G is a counter example. □

Claim 7.7. $\forall x \in N(v)$*, we have* $deg(x) \geq 3$*.*

Proof: Suppose not. Then by $Assumption$ 7.5, it is clear that the degree of the co-pivot,

$deg(u) = 2$. Let $N(u) = \{v, v'\}$. It is easy to verify from the description of configurations $B2 - B5$ and the fact that $deg(u) = 2$ that there can be at most two vertices in $N(v)$ whose degrees are greater than 3. By Claim 7.6, we know that $c'(u, v') \in F_v$. Let $D_v = D_v(c') = \{c'(v, x)|deg_G(x) \leq 3\}$. Clearly have $|D_v| \leq 2$.

If $c'(u, v') \in F_v - D_v$, then let $c = c'$. Else if $c'(u, v') \in D_v$, then recolor edge uv' using a color from $C - (S_{uv'} \cup D_v)$ to get a coloring c (Note that $|C - (S_{uv'} \cup D_v)| \geq \Delta + 3 - (\Delta - 1 + 2) = 2$ and since u' is a pendant vertex in $G - \{uu'\}$ the recoloring is valid). Now if $c(u, v') \notin F_v$, then it a contradiction to Claim 7.6. Thus $c(u, v') \in F_v - D_v$.

With respect to coloring c, let $c(u, v') = c(v, v_1)$. Now there are at least four candidate colors for the edge uv since $|F_u \cup F_v| \leq \Delta - 1$. If none of them are valid then they all have to be actively present in S_{vv_1}, implying that $|S_{vv_1}| \geq 4$, a contradiction since $|S_{vv_1}| \leq 3$. Thus there exists a color valid for the edge uv, a contradiction to the assumption that G is a counter example. □

Claim 7.8. $deg(v) > 3$. *Therefore* v *does not belong to* $Configuration\ B2$.

Proof: Suppose v belongs to $Configuration\ B2$. Let $N(v) = \{u, v_1, a\}$ such that $deg(u) \leq 4$ and $deg(v_1) \leq 4$. We also know from $Claim$ 7.7 that $deg(u) \geq 3$. Let $N(u) = \{x, y, v\}$, if $deg(u) = 3$ and let $N(u) = \{x, y, z, v\}$, if $deg(u) = 4$. Now the following cases occur:

- $|F_u \cap F_v| = 2$.
  Let $F_u \cap F_v = \{1, 2\}$. Also let $c(u, x) = c(v, a) = 1$ and $c(u, y) = c(v, v_1) = 2$. Since $|F_v \cup F_u| \leq 3$, there are at least Δ candidate colors for the edge vu. If none of them are valid then all those colors are actively present either in S_{vv_1} or S_{va}. Recalling that $|S_{va}| \leq \Delta - 1$ we can infer that there is at least one color $\alpha \in C - (F_v \cup F_u)$ that does not belong to S_{va}. Note that $|S_{vv_1} \cup F_v \cup F_u| \leq 6$ since $|S_{vv_1}| \leq 3$ and $|F_v \cup F_u| \leq 3$. Since $\Delta \geq 5$, we have $C - (S_{vv_1} \cup F_v \cup F_u) \neq \emptyset$. Recolor the edge vv_1 with the a color β from $C - (S_{vv_1} \cup F_v \cup F_u)$ to get a coloring c. The coloring c is valid because if a bichromatic cycle gets created due to recoloring then it has to be a $(\beta, 1)$ bichromatic cycle since $c(v, a) = 1$, implying that there existed a $(1, \beta, vv_1)$ critical path with respect to coloring c'. Recall that color β was not valid for the edge vu. Since $\beta \notin S_{vv_1}$, it implies that color β was actively present in S_{va}. This implies that there existed a $(1, \beta, vu)$ critical path with respect to coloring c'. Therefore by $Fact$ 2.1, there cannot exists a $(1, \beta, vv_1)$ critical path with respect to c', a contradiction. Thus the coloring c is valid. Now in c we have $F_v \cap F_u = \{1\}$ and $\alpha \notin S_{va}$. Thus color α is valid

for the edge vu, a contradiction to the assumption that G is a counter example.

- $|F_u \cap F_v| = 1$.

  Let $F_u \cap F_v = \{1\}$. Now if $c'(v, v_1) \in F_u \cap F_v$, then let $c'' = c'$. Otherwise let $c(u, x) = c(v, a) = 1$ and $c'(v, v_1) = 4$. If $deg(u) \leq 3$, then $|F_v \cup F_u| = 3$. Now there are at least Δ candidate colors for the edge vu. If none of them are valid then all the candidate colors are actively present in S_{va}, a contradiction since $|S_{va}| \leq \Delta - 1$. Thus there exists a valid color for the edge vu. Thus $deg(u) = 4$ and $|F_v \cup F_u| = 4$. Let $c(u, y) = 2$ and $c(u, z) = 3$. There are at least $\Delta - 1$ candidate colors for the edge vu. If none of them are valid then all the candidate colors are actively present in S_{va} and S_{ux}, implying that $S_{va} = S_{ux} = C - \{1, 2, 3, 4\}$. Now recolor edge ux using color 4 to get a coloring c''. It is valid by $Lemma$ 2.3 since $S_{ux} \cap S_{xu} = \emptyset$ (Note that $S_{xu}(c') = \{2, 3\}$).

  In both cases we have $\{c''(v, v_1)\} = F_u \cap F_v$. If none of the colors are valid for the edge vu, then all the candidate colors are actively present in $S_{vv'}$, implying that $S_{vv_1} = C - \{1, 2, 3, 4\}$. Since $\Delta \geq 5$, we have $|C - \{1, 2, 3, 4\}| \geq 8 - 4 = 4$. But $|S_{vv_1}| \leq 3$, a contradiction. Thus there exists a color valid for the edge vu, a contradiction to the assumption that G is a counter example.

□

In view of $Claim$ 7.8 we have $deg(v) > 3$. Therefore v belongs to configurations $B3$, $B4$ or $B5$. Now in view of $Claim$ 7.7, we have the following observation:

Observation 7.9. $deg(u) = 3$. *Let* $N(u) = \{v, w, z\}$.

In view of $Claim$ 7.6, we have the following two cases:

case 1: $|F_v \cap F_u| = 2$

Note that in this case $F_u \subseteq F_v$. Let $F_u = F_u \cap F_v = \{1, 2\}$. Let $c'(u, z) = 1$ and $c'(u, w) = 2$.

Claim 7.10. $F_u \nsubseteq F'_v$. *Therefore* $F''_v \cap F_u \neq \emptyset$.

Proof: Suppose not. Then let $c'(v, v_1) = c'(u, z) = 1$ and $c'(v, v_2) = c'(u, w) = 2$ (See the statement of $Lemma$ 7.4 for the naming convention of the neighbours of v). Since $|F_u \cup F_v| \leq \Delta - 1$, there are at least four candidate colors for the edge vu. If none of the candidate colors are valid for the edge vu, then we should have $S_{vv_1} \subset C - (F_u \cup F_v)$ and $S_{vv_2} \subset C - (F_u \cup F_v)$ since $|S_{vv_1}| = 2$ and $|S_{vv_2}| = 2$. Also $S_{vv_1} \cap S_{vv_2} = \emptyset$. Note that $C - (S_{vv_1} \cup F_v \cup F_u) \neq \emptyset$ since $|F_u \cup F_v| \leq \Delta - 1$ and $|S_{vv_1}| = 2$. Now assign a color from $C - (S_{vv_1} \cup F_u \cup F_v)$ to the edge vv_1 to get a coloring c. Recall that $S_{vv_1} \subset C - (F_u \cup F_v)$ and therefore $S_{vv_1} \cap S_{v_1 v} = \emptyset$. Thus by $Lemma$ 2.3, the coloring c is valid. With respect to the coloring c, $F_u \cap F_v = \{2\}$ and therefore if a candidate color is not valid for the edge vu, it has to be actively present in S_{vv_2}. Let $\alpha \in S_{vv_1}$. Clearly $\alpha \in C - (F_u \cup F_v)$ is a candidate color for the edge vu. Now since $\alpha \notin S_{vv_2}$ (recall that $S_{vv_1} \cap S_{vv_2} = \emptyset$), color α is valid for the edge vu, a contradiction to the assumption that G is a counter example. □

In view of $Claim$ 7.10, $F''_v \cap F_u \neq \emptyset$ and therefore $F''_v \neq \emptyset$. It follows that vertex v does not belong to configuration $B5$. Recalling $Claim$ 7.8, we infer that the vertex v belongs to either configuration $B3$ or $B4$. We take care of these two configurations separately below:

subcase 1.1: v belongs to configuration $B3$.
Since $deg(v) = 5$, we have $|F_v| = 4$. Let $F_v = \{1, 2, 3, 4\}$. Recall that by Claim 7.10, we have $F''_v \cap F_u \neq \emptyset$. Without loss of generality let $c'(u, z) = c'(v, a) = 1$ and $c'(u, w) = 2$. Now there are $\Delta - 1$ candidate colors for the edge vu. If none of them are valid then all these candidate colors are actively present in at least one of S_{uz} and S_{uw}. Let $Y = C - \{1, 2, 3, 4\}$. We make the following claim:

Claim 7.11. *With respect to any valid coloring c' of $G - \{uv\}$, $Y = S_{uz}$ and $Y = S_{uw}$.*
Proof: We use contradiction to prove the claim. Firstly we make the following subclaim:

subclaim 7.11.1: *With respect to any valid coloring c' of $G - \{uv\}$, one of S_{uz} or S_{uw} is Y.*
Proof: Suppose not. Then $Y \neq S_{uz}$ and $Y \neq S_{uw}$. Note that $|Y| = \Delta - 1$ while $|S_{uz}| \leq \Delta - 1$ and $|S_{uw}| \leq \Delta - 1$. Therefore there exist colors $\alpha, \beta \in Y$ such that $\alpha \notin S_{uz}$ and $\beta \notin S_{uw}$. Note that $\alpha \neq \beta$ since otherwise color $\alpha = \beta$ will be valid for the edge vu as there cannot exist a $(1, \alpha, vu)$ or $(2, \alpha, vu)$ critical path with respect to c'. It follows that α is actively present in S_{uw} and β is actively present in S_{uz}. Hence there exist $(2, \alpha, vu)$ and $(1, \beta, vu)$ critical paths. Now recolor edge uz using color α to get a coloring c''. The recoloring is valid since if there is a bichromatic cycle then it has to be a $(\alpha, 2)$ bichromatic cycle, implying that there existed

a $(2, \alpha, uz)$ critical path in c', a contradiction in view of Fact 2.1 as there already existed a $(2, \alpha, vu)$ critical path. With respect to coloring c'', $F_v \cap F_u = \{2\}$ and therefore if a candidate color is not valid for the edge vu, it has to be actively present in S_{uw}. Now color $\beta \notin S_{uw}$ and hence color β is valid for the edge vu, a contradiction to the assumption that G is a counter example. □

With respect to any valid coloring c' of $G - \{uv\}$, in view of $subclaim$ 7.11.1, let $u' \in \{w, z\}$ be such that $S_{uu'} = Y$. Let $\{u''\} = \{w, z\} - \{u'\}$. Now for contradiction assume that $S_{uu''} \neq Y$. Then clearly there exists a color $\alpha \in Y$ such that $\alpha \notin S_{uu''}$.

subclaim 7.11.2: *With respect to any valid coloring c' of $G - \{uv\}$, if exactly one of S_{uw} and S_{uz} is Y, say $S_{uu'} = Y$, then all the colors of Y are actively present in $S_{uu'}$ and $c'(u, u') \in F''_v$.*
Proof: Recolor the edge uu'' with the color α to get a coloring c''. Since $\alpha \notin S_{uu''}$ and α is not valid for the edge vu, color α is actively present in $S_{uu'}$ i.e., with respect to coloring c', there exists a (γ, α, vu) critical path, where $\gamma = c'(u, u')$. Thus by $Fact$ 2.1, there cannot exist a (γ, α, uu'') critical path and hence the coloring c'' is valid for the edge uu''. With respect to coloring c'', $F_v \cap F_u = \{2\}$. Now all the $\Delta - 2$ colors from $Y - \{\alpha\}$ are candidates for the edge vu. If any one of them is valid we are done. Thus none of them are valid and hence they all have to be actively present in $S_{uu'}$. Recalling that the color α was actively present in $S_{uu'}$ we infer that all the colors of Y are in fact actively present in $S_{uu'}$.

Now these colors will also be actively present in $S_{vv'}$, where $v' \in N(v)$ is such that $c'(v, v') = c'(u, u')$. This implies that $|S_{vv'}| = |Y| = \Delta - 1$. Therefore v' cannot be v_1 or v_2 since $|S_{vv_1}| = 2$ and $|S_{vv_2}| = 2$ while $\Delta - 1 \geq 4$. Thus $v' \in N''(v)$ implying that $c'(u, u') \in F''_v$. □

Recalling that for configuration $B3$, $|F''_v| = 2$ and since $1 \in F''_v$, at least one of $3, 4$ belongs to F'_v. Without loss of generality let $3 \in F'_v$. Now recolor edge uu' using color 3 to get a coloring d from c'. The coloring d is valid by $Lemma$ 2.3 since $\{d(u, u'')\} \cap S_{uu'} = \{2\} \cap Y = \emptyset$. With respect to the coloring d we have $S_{uu'} = Y$ and $S_{uu''} \neq Y$. Thus by $subclaim$ 7.11.2, $d(u, u') \in F''_v$, a contradiction since $d(u, u') = 3 \notin F''_v$. Thus we have $Y = S_{uz}$ and $Y = S_{uw}$. □

Since $Y = S_{uz}$ and $Y = S_{uw}$, we can recolor edge uz and uw using color from F'_v (Recall that with respect to configuration $B3$, $|F'_v| = 2$) to get a new valid coloring c. The coloring c is valid by $Lemma$ 2.3 since $F'_v \cap S_{uz} = F'_v \cap Y = \emptyset$ and $F'_v \cap S_{uw} = F'_v \cap Y = \emptyset$. This reduces the situation to $F_u \subseteq F'_v$, a contradiction to $Claim$ 7.10.

subcase 1.2: v belongs to configuration $B4$.

We have $deg(v) = 6$ and $F''_v = \{c'(v, a)\}$. Therefore in view of Claim 7.10, $c'(v, a)$ has to belong to F_u. Let $F_v = \{1, 2, 3, 4, 5\}$. Without loss of generality let $c'(u, w) = c'(v, v_1) = 2$ and $c'(u, z) = c'(v, a) = 1$. Now there are $\Delta - 2$ candidate colors for the edge vu. If none of them are valid then all these candidate colors are actively present in at least one of S_{uz} and S_{uw}. Let $X = C - \{1, 2, 3, 4, 5\}$.

Claim 7.12. $X \subseteq S_{uz}$.

Proof: Suppose not. Then let α be a color such that $\alpha \in X - S_{uz}$. This implies that α is actively present in S_{uw}. Hence there exists a $(2, \alpha, vu)$ critical path since $c'(u, w) = 2$. Now recolor edge uz using color α to get a coloring c''. The recoloring is valid since if there is a bichromatic cycle then it has to be a $(\alpha, 2)$ bichromatic cycle, implying that there existed a $(2, \alpha, uz)$ critical path in c', a contradiction in view of Fact 2.1 as there already existed a $(2, \alpha, vu)$ critical path. Now with respect to coloring c'', $F_v \cap F_u = \{2\}$ and therefore if none of the colors in $X - \{\alpha\}$ is valid for the edge vu, they all should be actively present in S_{uw}. Recalling that color α was actively present in S_{uw} we have all the colors of X actively present in S_{uw} and hence in S_{vv_1} implying that $|S_{vv_1}| \geq |X| = \Delta - 2 \geq 3$, a contradiction since $|S_{vv_1}| = 2$. Thus there exists a color valid for the edge vu, a contradiction to the assumption that G is a counter example. $\square$

Claim 7.13. $X \subseteq S_{uw}$.

Proof: Suppose not. Then let $X \nsubseteq S_{uw}$ and let α be a color such that $\alpha \in X - S_{uw}$. Recolor the edge uw using the color α. It is easy to see (by a similar argument used in the proof of Claim 7.12) that c'' is valid and all the colors of X are actively present in S_{uz} and hence in S_{va}.

Since $|X| = \Delta - 2$ and $|S_{va}| \leq \Delta - 1$, we have $|S_{va} - X| \leq 1$. If $S_{va} \neq X$, then the singleton set $S_{va} - X$ has to be a subset of $\{2, 3, 4, 5\}$ since $1 \notin S_{va}$. Without loss of generality let $S_{va} - X = \{2\}$ (Reader may note that $\{2, 3, 4, 5\} = F'_v$ and these four colors play symmetric roles in c'' and therefore we need to argue with respect to only one of them).

Recall that $c''(v, v_1) = c'(v, v_1) = 2$ and $|S_{vv_1}| = 2$. Of the colors 3, 4 and 5 let $3 \notin S_{vv_1}$. Also let $c''(v, v_2) = 3$. Now delete the color on the edge vv_2 and recolor the edge va using color 3 to get a coloring d. We claim that the coloring d is valid: If $S_{va} = X$, then clearly it is valid by *Lemma* 2.3 since $S_{va} \cap S_{av} = \emptyset$. Otherwise we have $S_{va} - X = \{2\}$ and if there is a bichromatic cycle with respect to the coloring d, it has to be a $(2, 3)$ bichromatic cycle. Since $d(v, v_1) = 2$, it means that $3 \in S_{vv_1}$, a contradiction to our assumption. Thus the coloring d is valid.

Now with respect to coloring d, we have $d(u, z) = 1$, $d(u, w) = \alpha$, $d(v, a) = 3$, $d(v, v_1) = 2$, $d(v, v_3) = 4$ and $d(v, v_4) = 5$. Edges vu and vv_2 are uncolored. Now let $X' = C - \{2, 3, 4, 5\}$. Note that $|X'| \geq 5$ since $\Delta \geq 6$. We show below that there exists a color in X' that is valid for the edge vv_2:

- $S_{vv_2} \subset X'$. Now any color in $X' - S_{vv_2}$ is valid for the edge vv_2 by *Lemma* 2.3.

- $|S_{vv_2} \cap X'| = 1$. In this case exactly one color, say $\theta \in \{2, 4, 5\}$ is present in S_{vv_2} since $3 \notin S_{vv_2}$ (This is because $c'(v, v_2) = 3$). Now there are at least four candidate colors for the edge vv_2 since $|F_v \cup F_u| \leq 4+2-1 = 5$ and there are at least $\Delta+3 \geq deg(v)+3 = 6+3 = 9$ colors in C. If none of the candidate colors are valid then a (θ, γ) bichromatic cycle should form for each $\gamma \in X' - S_{vv_2}$. Since $\theta \in \{2, 4, 5\}$, we have $\theta = d(v, v_j)$ for $j = 1$, 3 or 4. It means that each of the (θ, γ) bichromatic cycle should contain the edge vv_j and thus $X' - S_{vv_2} \subseteq S_{vv_j}$. But $|X' - S_{vv_2}| \geq 5 - 2 + 1 \geq 4$ and $|S_{vv_j}| = 2$, a contradiction. Thus at least one color will be valid for the edge vv_2.

- $S_{vv_2} \cap X' = \emptyset$. Now all the colors in X' are candidates for the edge vv_2. If none of them are valid then all these candidate colors have to form bichromatic cycles with at least one of the colors in $S_{vv_2} \cap F_v$. Now since $c''(v, v_2) = 3$, color $3 \notin S_{vv_2}(d)$ and therefore 3 is not involved in any of these bichromatic cycles. Also since $|S_{vv_2}| = 2$, exactly two of the colors from $\{2, 4, 5\}$ and hence exactly two of the edges from $\{vv_1, vv_3, vv_4\}$ are involved in these bichromatic cycles. But we know that $|S_{vv_1}| = |S_{vv_3}| = |S_{vv_4}| = 2$. It follows that at most four bichromatic cycles can be formed. But $|X'| \geq 5$ and thus at least one color will be valid for the edge vv_2.

Let $\beta \in X'$ be a valid color for vv_2. Color the edge vv_2 using β to get a new coloring d'. Now:

- If $\beta \in C - \{1, 2, 3, 4, 5, \alpha\}$, then $F_v \cap F_u = \emptyset$ with respect to d', a contradiction to *Claim* 7.6.

- If $\beta \in \{1, \alpha\}$, then there are at least three candidate colors for the edge vu since $\Delta \geq 6$. Moreover we have $F_v \cap F_u = \{\beta\}$. If none of these three candidate colors are valid for the edge vu, then all of them have to be actively present in S_{vv_2}, implying that $|S_{vv_2}| \geq 3$, a contradiction since $|S_{vv_2}| = 2$. Therefore at least one of the three candidate colors is valid for the edge vu.

Thus we have a valid color for edge vu, a contradiction to the assumption that G is a counter example. □

In view of $Claim$ 7.12, $Claim$ 7.13 and from $|S_{uz}|$, $|S_{uw}| \leq \Delta - 1$ and $|X| = \Delta - 2$, it is easy to see that $|(S_{uz} \cup S_{uw}) - X| \leq 2$. Thus recalling that $3, 4, 5 \notin X$, we infer that $\{3, 4, 5\} - (S_{uz} \cup S_{uw}) \neq \emptyset$. Now recolor the edge uz using a color $\mu \in \{3, 4, 5\} - (S_{uz} \cup S_{uw})$. Clearly μ is a candidate for the edge uz since $d'(u, w) = 2$ and $\mu \notin S_{uz}$. Moreover μ is valid for uz since if otherwise a $(2, \mu)$ bichromatic cycle has to be formed containing uw, implying that $\mu \in S_{uw}$, a contradiction. This reduces the situation to $F_u \subseteq F'_v$, a contradiction to $Claim$ 7.10.

case 2: $|F_v \cap F_u| = 1$

Recall that by $Claim$ 7.8 and $Claim$ 7.7, v belongs to configurations $B3$, $B4$ or $B5$ and $deg(u) = 3$. As before $N(u) = \{v, w, z\}$. Also let $F_v \cap F_u = \{1\}$.

Claim 7.14. *With respect to any valid coloring of* $G - \{vu\}$, $F_u \cap F'_v = \emptyset$. *This implies that* $F_v \cap F_u \subseteq F''_v$.

Proof: Suppose not. Then without loss of generality let $c'(v, v_1) = c'(u, z) = 1$. Recalling $deg(u) = 3$, $|F_u| \leq 2$ and thus $|F_u \cup F_v| \leq (\Delta - 1) + 2 - 1 = \Delta$. It follows that there are at least three candidate colors for the edge vu. If none of the candidate colors are valid for the edge vu, then all these candidate colors have to be actively present in S_{vv_1}, implying that $|S_{vv_1}| \geq 3$, a contradiction since $|S_{vv_1}| = 2$. It follows that at least one of the three candidate colors is valid for the edge vu, a contradiction to the assumption that G is a counter example. □

In view of $Claim$ 7.14, $F''(v) \neq \emptyset$ and therefore the vertex v cannot belong to configuration $B5$. We infer that v has to belong to either configuration $B3$ or $B4$. We take care of these two subcases separately below:

subcase 2.1: v **belongs to configuration** $B3$**.**

Since $deg(v) = 5$, we have $|F_v| = 4$. Let $F_v \cup F_u = \{1, 2, 3, 4, 5\}$. By Claim 7.14, we have $F_v \cap F_u = \{1\} \subseteq F''_v = \{c'(v, a), c'(v, b)\}$. Without loss of generality let $c'(u, z) = c'(v, a) = 1$. Also let $c'(u, w) = 2$, $c'(v, b) = 3$, $c'(v, v_1) = 4$ and $c'(v, v_2) = 5$. Since $|F_v \cup F_u| = 5$, there are $\Delta - 2$ candidate colors for the edge vu. If none of them are valid then there exists a $(1, \alpha, vu)$ critical path for each $\alpha \in C - (F_v \cup F_u) = C - \{1, 2, 3, 4, 5\}$. Thus we have the following observation:

Observation 7.15. *With respect to the coloring* c'*, each color in* $C - \{1, 2, 3, 4, 5\}$ *is actively present in* S_{uz} *as well as* S_{va}*.*

Claim 7.16. $S_{uz} = C - \{1, 3, 4, 5\}$ *and* $1, 4, 5 \in S_{uw}$*.*
Proof: Since $C - \{1, 2, 3, 4, 5\} \subseteq S_{uz}$ and $|S_{uz} - (C - \{1, 2, 3, 4, 5\})| \leq 1$ we infer that at most one of 4, 5 can be present in S_{uz}. Suppose one of 4, $5 \in S_{uz}$. Without loss of generality let $4 \in S_{uz}$. Now recolor edge uz using color 5. It is valid by $Lemma$ 2.3 since $S_{uz} \cap S_{zu} = S_{uz} \cap \{2\} = \emptyset$. Thus we have reduced the situation to $F_u \cap F'_v \neq \emptyset$, a contradiction to $Claim$ 7.14. Thus we have $4, 5 \notin S_{uz}$. Recolor edge uz using color 4 or 5. If any one of them is valid then we will have $F_u \cap F'_v \neq \emptyset$ with respect to this new coloring, a contradiction to $Claim$ 7.14. It follows that none of them are valid. That is, bichromatic cycles get formed due to the recoloring. Clearly the bichromatic cycles have to be $(2, 4)$ and $(2, 5)$ bichromatic cycles since $c'(u, w) = 2$. Thus $2 \in S_{uz}$ and $4, 5 \in S_{uw}$. Recalling that $C - \{1, 2, 3, 4, 5\} \subseteq S_{uz}$ and $|S_{uz}| \leq \Delta - 1$ we can infer that $S_{uz} = C - \{1, 3, 4, 5\}$.

Now if $1 \notin S_{uw}$, then assign color 1 to edge uw and the color 4 to edge uz. Clearly this recoloring is valid by $Lemma$ 2.3 since $S_{zu} \cap S_{uz} = \{1\} \cap C - \{1, 3, 4, 5\} = \emptyset$. With respect to the new coloring, $F_u \cap F_v = \{1, 4\}$ which reduces the situation to $case$ 1. Thus we infer that $1 \in S_{uw}$. Therefore we have $1, 4, 5 \in S_{uw}$. □

Claim 7.17. $|(C - \{1, 2, 3, 4, 5\}) - S_{uw}| \geq 2$.
Proof: Since $|S_{uw}| \leq \Delta - 1$ there are at least four colors missing from S_{uw}. Thus even if colors 2 and 3 are missing from S_{uw} there should be at least two colors in $C - \{1, 2, 3, 4, 5\}$

that are absent in S_{uw} since $1, 4, 5 \in S_{uw}$ by $Claim$ 7.16. □

Now discard the color on the edge uw to obtain a partial coloring d of G from c'.

Claim 7.18. *With respect to coloring d, $\forall \alpha \in C - \{1, 3, 4, 5\}$, there exists a $(1, \alpha, vu)$ critical path.*

Proof: With respect to the coloring c', there existed $(1, \alpha, vu)$ critical path for all $\alpha \in C - (F_v \cup F_u) = C - \{1, 2, 3, 4, 5\}$ by $Observation$ 7.15. These critical paths remain unaltered when we get d from c'. Thus these critical paths are present in d also. Thus it is enough to prove that there exists $(1, 2, vu)$ critical path with respect to the coloring d. Let $\theta \in (C - \{1, 2, 3, 4, 5\}) - S_{uw}$. Note that θ exists by $Claim$ 7.17. Now color θ is a candidate for the edge uw since $\theta \notin S_{uw}$ and $d(u, z) = 1$. Recolor the edge uw using color θ to get a coloring d'. The coloring d' is valid since otherwise a $(1, \theta)$ bichromatic cycle has to be created due to the recoloring. This means that there existed a $(1, \theta, uw)$ critical path with respect to coloring c', a contradiction by $Fact$ 2.1 as there already existed a $(1, \theta, vu)$ critical path with respect to the coloring c' by $Observation$ 7.15. Thus the coloring d' is valid.

Now color 2 is a candidate for the edge vu. If it is valid we get a valid coloring for G. Thus it is not valid. This means that there exists a $(1, 2, vu)$ critical path with respect to the coloring d' since $F_v \cap F_u = \{1\}$ with respect to the coloring d'. Now it is easy to see that this $(1, 2, vu)$ critical path will also exist with respect to coloring d. Thus with respect to the coloring d, $\forall \alpha \in C - \{1, 3, 4, 5\}$, there exists a $(1, \alpha, vu)$ critical path. □

Observation 7.19. *Let $Q = (C - \{1, 3, 4, 5\}) - S_{uw}$. From Claim 7.17, we know that $|(C - \{1, 2, 3, 4, 5\}) - S_{uw}| \geq 2$. Since $c'(u, w) = 2$ we have $2 \notin S_{uw}$. From this we can infer that $2 \in Q$. Thus $|Q| \geq 3$.*

Claim 7.20. *There exists a color $\gamma \in Q$ such that γ is valid for the edge vv_1 or vv_2.*

Proof:

Recall that $|S_{vv_1}| = 2$, $|S_{vv_2}| = 2$ and by $Observation$ 7.15, $|Q| \geq 3$.

- If $S_{vv_1} \subset Q$ or $S_{vv_2} \subset Q$. Without loss of generality let $S_{vv_1} \subset Q$. Let γ be a color in $Q - S_{vv_1}$. Recolor edge vv_1 using color γ to get a coloring d'. The coloring d' is valid by $Lemma$ 2.3 as $S_{vv_1} \cap S_{v_1v} = \emptyset$ since $Q \cap F_v = \emptyset$.

- If $S_{vv_1} \nsubseteq Q$ and $S_{vv_2} \nsubseteq Q$. In this case, at most one color in Q can be in S_{vv_1} and the same holds true for S_{vv_2}. Thus all the colors of Q except for one are candidates for edge vv_1 and all the colors of Q except for one are candidates for edge vv_2. Since $|Q| \geq 3$, we can infer that there exists a color $\gamma \in Q$ which is a candidate for both vv_1 and vv_2.

subclaim *Color γ is valid either for the edge vv_1 or for the edge vv_2.*

Proof: Recolor vv_1 using color γ. If γ is valid, we are done. If it is not valid, then there has to be a (γ, θ) bichromatic cycle getting formed, where $\theta \in F_v - \{d(v, v_1)\} = F_v - \{4\} = \{1, 3, 5\}$. But this cannot be a $(\gamma, 5)$ bichromatic cycle since $\gamma \notin S_{vv_2}$ (recall that $d(v, v_2) = c'(v, v_2) = 5$). Also this cannot be a $(\gamma, 1)$ bichromatic cycle since otherwise it implies that there exists a $(1, \gamma, vv_1)$ critical path with respect to the coloring d, a contradiction in view of $Fact$ 2.1 as there already exists a $(1, \gamma, vu)$ critical path by $Claim$ 7.18. Thus it has to be a $(3, \gamma)$ bichromatic cycle, implying that there existed a $(3, \gamma, vv_1)$ critical path with respect to the coloring d.

If γ is not valid for the edge vv_1 we recolor edge vv_2 instead, using color γ to get a coloring d' form d. We claim that the coloring d' is valid. This is because there cannot be a $(\gamma, 4)$ bichromatic cycle since $\gamma \notin S_{vv_1}$ (recall that $d(v, v_1) = c'(v, v_1) = 4$). Also there cannot be a $(\gamma, 1)$ bichromatic cycle since otherwise it implies that there exists a $(1, \gamma, vv_2)$ critical path with respect to the coloring d, a contradiction in view of $Fact$ 2.1 as there already exists a $(1, \gamma, vu)$ critical path by $Claim$ 7.18. Finally there cannot be a $(3, \gamma)$ bichromatic cycle because this implies that there existed a $(3, \gamma, vv_2)$ critical path with respect to the coloring d, a contradiction by $Fact$ 2.1 since there already existed a $(3, \gamma, vv_1)$ critical path with respect to the coloring d. Thus the coloring d' is valid. □

□

In view of Claim 7.20, without loss of generality let $\gamma \in Q$ be valid for the edge vv_1. Now we recolor the edge vv_1 using color γ to get a coloring d'.

We claim that none of the colors in S_{uw} were altered in this recoloring. This is because if they are altered then vv_1 has to be an edge incident on w and thus one of the end points of vv_1 has to be w. Since v cannot be w, either v_1 should be w. But we know that $deg(v_1) = 3$. Recall that $1, 4, 5 \in S_{uw}$ and thus $deg(w) \geq 4$. Thus v_1 cannot be w. Thus none of the colors of S_{uw}

are modified while getting d' from d. We infer that $\gamma \notin S_{uw}$ since $Q \cap S_{uw} = \emptyset$. Therefore γ is a candidate for the edge uw since $d'(u,z) = 1$. Now color the edge uw using the color γ to get a coloring d''. If the coloring d'' is valid, then we have $F_u \cap F_v = \{1, \gamma\}$. This reduces the situation to $case$ 1.

On the other hand if the coloring d'' is not valid then there has to be a bichromatic cycle formed due to the recoloring of edge uw. Since $d''(u,z) = 1$, it has to be a $(1,\gamma)$ bichromatic cycle. Recall that there existed a $(1,\gamma,vu)$ critical path with respect to the coloring d. Note that to get d'' from d we have only recolored two edges namely vv_1 and uw, both with color γ. Clearly these recolorings cannot break the $(1,\gamma,vu)$ critical path that existed in d, but only can extend it. Thus we can infer that in d'' the $(1,\gamma)$ bichromatic cycle passes through v and hence through the edges va and vv_1. Now recolor edge va using color 4 to get a coloring c. Recall that $S_{va} = C - \{1,3,4,5\}$ by Claim 7.18 and $S_{av} = F_v - \{c''(v,a)\} = \{3,5,\gamma\}$. Therefore color 4 is indeed a candidate for edge va. Note that by recoloring va using color 4, we have broken the $(1,\gamma)$ bichromatic cycle that existed in d''. Now we claim that the coloring c is valid. Note that $S_{va} \cap S_{av} = S_{va} \cap \{3,5,\gamma\} = \{\gamma\}$. If a bichromatic cycle gets formed due to this recoloring then it has to be $(4,\gamma)$ bichromatic cycle, implying that $4 \in S_{vv_1}$. But $S_{vv_1}(c) = S_{vv_1}(d'') = S_{vv_1}(d)$ and $4 \notin S_{vv_1}(d)$ since $d(v,v_1) = 4$. Thus $4 \notin S_{vv_1}(c)$, a contradiction. Thus the coloring c is valid. With respect to the coloring c, we have $F_v \cap F_u = \{\gamma\} \subset F'_v$, a contradiction to $Claim$ 7.14.

subcase 2.2: v **belongs to configuration** $B4$**.**

We have $deg(v) = 6$ and therefore $|F_v| = 5$. Moreover $|F''_v| = 1$ and $|F'_v| = 4$. By $Claim$ 7.14, $F_v \cap F_u = \{1\} \subseteq F''_v$. Without loss of generality let $c'(u,z) = c'(v,a) = 1$. Also let $c(u,w) = 2$, $F'_v = \{3,4,5,6\}$ and $Z = \{3,4,5,6\}$. There are $\Delta - 3$ candidate colors for the edge vu. If none of them are valid then there exist $(1,\alpha,vu)$ critical path for each $\alpha \in C - (F_v \cup F_u) = C - \{1,2,3,4,5,6\}$. Thus we have the following observation:

Observation 7.21. *With respect to the coloring c', each color in $C - \{1,2,3,4,5,6\}$ is actively present in S_{uz} as well as S_{va}.*

Claim 7.22. $S_{uz} \supseteq C - \{1,3,4,5,6\}$ *and* $1 \in S_{uw}$. *Also at least three of the colors from Z are present in S_{uw}.*

Proof: As we have seen above $C - \{1,2,3,4,5,6\} \subseteq S_{uz}$. Suppose $2 \notin S_{uz}$. Note that every color in $C - (S_{uz} \cup S_{zu})$ is a candidate for uz. Now $S_{zu} = \{c'(u,w)\} = \{2\}$. Moreover $|S_{uz}| \leq \Delta - 1$ and thus S_{uz} can have at most two more colors other than those in $C - \{1,2,3,4,5,6\}$. From this we can infer that at least two of the colors in Z are candidates for the edge uz. They are also valid by $Lemma$ 2.3 since $S_{uz} \cap S_{zu} = S_{uz} \cap \{2\} = \emptyset$. Thus we can reduce the situation to $F_u \cap F'_v \neq \emptyset$, by assigning one of the valid colors from Z to uz, thereby getting a contradiction to $Claim$ 7.14. Thus we infer that $2 \in S_{uz}$. Therefore we get $S_{uz} \supseteq C - \{1,3,4,5,6\}$. Since $|S_{uz}| \leq \Delta - 1$ and $|C - \{1,3,4,5,6\}| = \Delta - 2$ we can infer that $|Z \cap S_{uz}| \leq 1$.

If any one of the colors in $Z - S_{uz}$ is valid for the edge uz, then it will reduce the situation to $F_u \cap F'_v \neq \emptyset$, a contradiction to $Claim$ 7.14. Thus none of these colors are valid for the edge uz. Therefore there should be bichromatic cycles getting formed when we try to recolor edge uz using any of these colors. These bichromatic cycles have to be $(2,\mu)$ bichromatic cycles for each color $\mu \in Z - S_{uz}$ since $c'(u,w) = 2$. Thus we can infer that at least three of the colors from Z are present in S_{uw} since $|Z - S_{uz}| \geq 4 - 1 = 3$.

Now if $1 \notin S_{uw}$, then assign color 1 to edge uw and a color $\mu \in Z - S_{uz}$ to edge uz. Clearly this recoloring is valid by $Lemma$ 2.3 since $S_{zu} \cap S_{uz} = \{1\} \cap S_{uz} = \emptyset$ ($1 \notin S_{uz}$ since $c'(u,z) = 1$). With respect to the new coloring, $F_u \cap F_v = \{1, \mu\}$ which reduces the situation to $case$ 1. Thus we infer that $1 \in S_{uw}$. □

Claim 7.23. $|(C - \{1,2,3,4,5,6\}) - S_{uw}| \geq 2$.

Proof: Since $|S_{uw}| \leq \Delta - 1$, we have $|C - S_{uw}| \geq 4$. Now since $|Z \cap S_{uw}| \geq 3$ and $1 \in S_{uw}$, $|\{1,2,3,4,5,6\} \cap S_{uw}| \geq 4$. It follows that $|(C - S_{uw}) \cap \{1,2,3,4,5,6\}| \leq 2$ and the Claim follows. □

Now discard the color on the edge uw to obtain a partial coloring d of G from c'.

Claim 7.24. *With respect to coloring* d, $\forall \alpha \in C - \{1,3,4,5,6\}$, *there exists a* $(1,\alpha,vu)$ *critical path and thus* $C - \{1,3,4,5,6\} \subseteq S_{va}$.

Proof: With respect to the coloring c', there existed a $(1,\alpha,vu)$ critical path for each $\alpha \in C - (F_v \cup F_u) = C - \{1,2,3,4,5,6\}$ by $Observation$ 7.21. These critical paths remain unaltered when we get d from c'. Thus these critical paths are present in d also. Thus it is enough to prove that there exists a $(1,2,vu)$ critical path with respect to the coloring d. Let $\theta \in (C - \{1,2,3,4,5,6\}) - S_{uw}$. Note that θ exists by $Claim$ 7.23. Now color θ is a candidate for the edge uw since $\theta \notin S_{uw}$ and $d(u, z) = 1$. Recolor the edge uw using color θ to get a coloring d'. The coloring d' is valid since otherwise a $(1, \theta)$ bichromatic cycle has to be created due to the recoloring. This means that there existed a $(1, \theta, uw)$ critical path with respect to coloring c', a contradiction by $Fact$ 2.1 as there already existed a $(1, \theta, vu)$ critical path with respect to the coloring c' by $Observation$ 7.21. Thus the coloring d' is valid.

Now color 2 is a candidate for the edge vu. If it is valid we get a valid coloring for G. Thus it is not valid. This means that there exists a $(1, 2, vu)$ critical path with respect to the coloring d' since $F_v \cap F_u = \{1\}$ with respect to the coloring d'. Now it is easy to see that this $(1, 2, vu)$ critical path will also exist with respect to coloring d. Thus with respect to the coloring d, $\forall \alpha \in C - \{1, 3, 4, 5, 6\}$, there exists a $(1, \alpha, vu)$ critical path. □

Observation 7.25. *Let $Q = (C - \{1, 3, 4, 5, 6\}) - S_{uw}$. From Claim 7.23, we know that $|(C - \{1, 2, 3, 4, 5, 6\}) - S_{uw}| \geq 2$. Since $c'(u, w) = 2$ we have $2 \notin S_{uw}$. From this we can infer that $2 \in Q$. Thus $|Q| \geq 3$.*

Recall that $|S_{vv_i}| = 2$, for $i \in \{1, 2, 3, 4\}$ and by $Observation$ 7.25, $|Q| \geq 3$. We know that $S_{va} \supseteq C - \{1, 3, 4, 5, 6\}$ by $Claim$ 7.24. Since $|C - \{1, 3, 4, 5, 6\}| = \Delta - 2$ and $|S_{va}| \leq \Delta - 1$ we have $|Z \cap S_{va}| = |\{3, 4, 5, 6\} \cap S_{va}| \leq 1$. We make the following assumption:

Assumption 7.26. *If $Z \cap S_{va} \neq \emptyset$, let $\{\alpha\} = Z \cap S_{va}$ and let $d(v, v_t) = \alpha$, where $t \in \{1, 2, 3, 4\}$. Let $\beta \in (Z - \{\alpha\}) - S_{vv_t}$. If $Z \cap S_{va} = \emptyset$, then let β be any color in Z.*

We now plan to recolor one of the edges in $\{vv_1, vv_2, vv_3, vv_4\}$ using a specially selected color $\gamma \in Q$. After this we will also use the same color γ to recolor edge uw, with the intention of reducing the situation to $case$ 1. Below we give the recoloring procedure for the rest of the proof starting from the current coloring d in 3 steps. The final coloring c of $G - \{vu\}$ that we obtain at the end of $Step 3$ will give the required contradiction.

Step1: With respect to the coloring d,

(i) **If one of the edges vv_i, for $i \in \{1, 2, 3, 4\}$ is such that $S_{vv_i} \subset Q$, then recolor that edge with any color $\gamma \in Q - S_{vv_i}$. We call the edge that we chose to recolor as $(v, v_{t'})$.**

(ii) **If $\forall i \in \{1, 2, 3, 4\}$, $S_{vv_i} \nsubseteq Q$, then we select an edge $vv_{t'}$, where $t' \in \{1, 2, 3, 4\}$ such**

that $d(v, v_{t'}) = \beta$ (See $Assumption$ 7.26). Now recolor the edge $vv_{t'}$ with a suitably selected (see the proof of $Claim$ 7.27) color in $Q - S_{vv_{t'}}$.

The resulting coloring after performing $Step1$ is named d'.

Claim 7.27. *There exists a color $\gamma \in Q$ such that the coloring d' obtained after $Step1$ is valid.*
Proof: At the beginning of $Step1$, we had the following possible cases:

(i) **One of the edges vv_i, for $i \in \{1, 2, 3, 4\}$ is such that $S_{vv_i} \subset Q$:**
Let γ be a color in $Q - S_{vv_i}$. Recolor edge vv_i using color γ to get a coloring d'. The coloring d' is valid by $Lemma$ 2.3 as $S_{vv_i} \cap S_{v_iv} = \emptyset$ since $Q \cap F_v = \emptyset$.

(ii) **$S_{vv_i} \not\subseteq Q$, for each $i \in \{1, 2, 3, 4\}$:**
Let t' be as defined in $Step1$. Clearly all the colors in $Q - S_{vv_{t'}}$ are candidates for $vv_{t'}$ since $Q \cap F_v = \emptyset$. Note that since $S_{vv_i} \not\subseteq Q$ we have $|Q \cap S_{vv_{t'}}| \leq 1$ and therefore $|Q - S_{vv_{t'}}| \geq 2$. If any one of the candidate colors is valid for the edge $vv_{t'}$, the statement of the Claim is obviously true. On the other hand if none of them are valid, then there has to be a (γ, θ) bichromatic cycle getting formed, for some $\theta \in F_v - \{d(v, v_{t'})\} = F_v - \{\beta\}$ when we try to recolor edge $vv_{t'}$ using color γ, for each $\gamma \in Q - S_{vv_{t'}}$. Note that $\theta \neq 1$ because if a $(\gamma, 1)$ bichromatic cycle gets formed, then there has to be a $(1, \gamma, vv_{t'})$ critical path with respect to the coloring d, a contradiction in view of $Fact$ 2.1 as there already exists a $(1, \gamma, vu)$ critical path by $Claim$ 7.24. Thus $\theta \in F'_v - \{d(v, v_{t'})\}$ since $F''_v = \{1\}$. Therefore we have $|(F'_v - \{d(v, v_{t'})\}) \cap S_{vv_{t'}}| \geq 1$. We have the following cases:

- **$|(F'_v - \{d(v, v_{t'})\}) \cap S_{vv_{t'}}| = 1$:** Let $S_{vv_{t'}} \cap (F'_v - \{d(v, v_{t'})\}) = d(v, v')$, for $v' \in \{v_1, v_2, v_3, v_4\} - \{v_{t'}\}$. Thus all the candidate colors of $vv_{t'}$, namely all the colors of $Q - S_{vv_{t'}}$ should form bichromatic cycles passing through the edge vv', implying that $Q - S_{vv_{t'}} \subset S_{vv'}$. But $|Q - S_{vv_{t'}}| \geq 2$ and $|S_{vv'}| = 2$. Thus $S_{vv'} = Q - S_{vv_{t'}} \subseteq Q$, a contradiction.
- **$|(F'_v - \{d(v, v_{t'})\}) \cap S_{vv_{t'}}| = 2$:** This means that $S_{vv_{t'}} \subseteq F'_v$ and therefore we have $Q \cap S_{vv_{t'}} = \emptyset$. Thus $|Q - S_{vv_{t'}}| = |Q| \geq 3$. Therefore there are at least three candidate colors for the edge $vv_{t'}$. Let $S_{vv_{t'}} \cap (F'_v - \{d(v, v_{t'})\}) = \{d(v, v'), d(v, v'')\}$, for $v', v'' \in \{v_1, v_2, v_3, v_4\} - \{v_{t'}\}$. Since for each candidate color we have a bichromatic cycle, we can infer that there are at least three bichromatic cycles, each of

them passing through either vv' or vv''. Thus at least two bichromatic cycles have to pass through one of vv' and vv''. But since $|S_{vv'}| = 2$ and $|S_{vv''}| = 2$, we can infer that either $S_{vv'} \subseteq Q$ or $S_{vv''} \subseteq Q$, a contradiction.

□

Step2: Let γ be the color which was used to recolor the edge $vv_{t'}$ in $Step1$. Now recolor edge uw with color γ to get a coloring d''.

Claim 7.28. *The coloring d'' is proper.*

Proof: We claim that none of the colors in S_{uw} were altered in $Step1$. This is because if they are altered then the edge $vv_{t'}$ should be incident on w and thus one of the end points of $vv_{t'}$, where $t' \in \{1, 2, 3, 4\}$, has to be w. Since v cannot be w, $v_{t'}$ should be w. But we know that $deg(v_i) = 3$. Recall that $|Z \cap S_{uw}| \geq 3$ by $Claim$ 7.22 and thus $|S_{uw}| \geq 3$. Therefore $deg(w) \geq 4$. Thus $v_{t'}$ cannot be w. Thus none of the colors of S_{uw} are modified while getting d' from d. Recall that $Q = (C - \{1, 3, 4, 5, 6\}) - S_{uw}$ and thus $\gamma \notin S_{uw}$. Therefore γ is a candidate for the edge uw since $d(u, z) = 1$. Thus the coloring d'' is proper. □

If the coloring d'' is valid, then we have $F_u \cap F_v = \{1, \gamma\}$ for a valid coloring of $G - \{vu\}$. This reduces the situation to $case$ 1. Thus coloring d'' is not valid. Since the coloring d'' is not valid, there has to be a bichromatic cycle formed due to the recoloring of edge uw. Since $d''(u, z) = 1$, it has to be a $(1, \gamma)$ bichromatic cycle. Recall that there existed a $(1, \gamma, vu)$ critical path with respect to the coloring d by $Claim$ 7.24. Note that to get d'' from d we have only recolored two edges namely $vv_{t'}$ and uw, both with color γ. Clearly these recolorings cannot break the $(1, \gamma, vu)$ critical path that existed in d, but can only extend it. Thus we can infer that in d'' the $(1, \gamma)$ bichromatic cycle passes through v and hence through the edges va and $vv_{t'}$. Also note that this can happen only when we have $1 \in S_{vv_{t'}}$. Thus $S_{vv_{t'}} \nsubseteq Q$. It means that substep (ii) of $Step1$ was executed; and the color on $vv_{t'}$ with respect to coloring d was β (from $Assumption$ 7.26). We break the $(1, \gamma)$ bichromatic cycle as follows:

Step3: Recolor the edge va with color β (see in $Assumption$ 7.26) to get a coloring c.

Claim 7.29. *The coloring c is valid.*

Proof: Recall by $Assumption$ 7.26 that $\beta \notin S_{va}$. Also clearly $\beta \notin F_v(d'')$ since we recolored $vv_{t'}$ by a color $\gamma \in Q$ to get d'' form d ($\beta \neq \gamma$ since $\beta \in F_v(d)$ and $F_v(d) \cap Q = \emptyset$). Therefore color β is a candidate for edge va. Note that by recoloring va using color β, we have broken the $(1, \gamma)$ bichromatic cycle that existed in d''. We claim that the coloring c is valid. Otherwise there has to be a bichromatic cycle involving β and a color in $S_{va} \cap S_{av}$. But $S_{av} = (Z - \{\beta\}) \cup \{\gamma\} = (\{3, 4, 5, 6\} - \{\beta\}) \cup \{\gamma\}$. Since with respect to d'' there was a $(1, \gamma)$ bichromatic cycle passing through the edges va and $d''(v, a) = 1$, we have $\gamma \in S_{va} \cap S_{av}$. But there cannot be a (β, γ) bichromatic cycle getting formed in c since such a cycle should contain edge $vv_{t'}$ and thus $\beta \in S_{vv_{t'}}$. But $S_{vv_{t'}}(c) = S_{vv_{t'}}(d'') = S_{vv_{t'}}(d)$ and $\beta \notin S_{vv_{t'}}(d)$ since $d(v, v_{t'}) = \beta$. Thus $\beta \notin S_{vv_1}(c)$, a contradiction. Thus there cannot be a (β, γ) bichromatic cycle.

Thus if the coloring c is not valid then there has to be a bichromatic cycle involving β and one of the colors in $Z - \{\beta\} \cap S_{va}$. We know by $Assumption$ 7.26 that $Z \cap S_{va} = \alpha$. Thus it has to be a (β, α) bichromatic cycle. Since $c(v, v_t) = d(v, v_t) = \alpha$, this bichromatic cycle contains the edge vv_t and hence $\beta \in S_{vv_t}$, a contradiction to the way β was selected in $Assumption$ 7.26. Thus there cannot be a (β, α) bichromatic cycle. Thus the coloring c is valid. □

With respect to the coloring c, we have $F_v \cap F_u = \{\gamma\} \subset F'_v$, a contradiction to $Claim$ 7.14.

7.2.2 There exists no vertex v that belongs to one of the configurations $B2$, $B3$, $B4$ or $B5$

This means that there exists a vertex v that belongs to configuration $B1$, i.e., $deg(v) = 2$. Let $Q = \{u \in V \ : \ deg(u) = 2\}$. First we claim that Q is an independent set in G. Otherwise let $u', u \in Q$ be such that $(u, u') \in E(G)$. Now since G is a minimum counter example, $G - \{uu'\}$ is acyclically edge colorable using $\Delta + 3$ colors. Let c' be a valid coloring of $G - \{uu'\}$. Now if $F_u \cap F_{u'} = \emptyset$, then there are $\Delta + 3 - 2 = \Delta + 1$, candidate colors for the edge uu'. Since $S_{uu'} \cap S_{u'u} = \emptyset$, by $Lemma$ 2.3, all the candidate colors are valid for the edge uu'. On the other hand if $|F_u \cap F_{u'}| = 1$, then there are $\Delta + 3 - 1 = \Delta + 2$ candidate colors for the edge uu'. Let $N(u) = \{u', u''\}$. If none of them are valid then all those candidate colors have to be actively present in $S_{uu''}$, implying that $|S_{uu''}| \geq \Delta + 2$, a contradiction since $|S_{uu''}| \leq \Delta - 1$. Thus there exists a valid coloring of G, a contradiction to the assumption that G is a counter example. We infer that Q is an independent set in G.

Now delete all the vertices in Q from G to get a graph G'. Clearly the graph G' has at most $2|V(G')| - 1$ edges since Q is an independent set. It follows by $Lemma$ 7.4 that there should be a vertex v' in G' such that v' is the pivot of one of the configurations $B1 - B5$, say $B' = \{v'\} \cup N_{G'}(v')$. But with respect to graph G, $\{v'\} \cup N_{G'}(v')$ did not form any of the configurations $B1 - B5$. This means that the degree of at least one of the vertices in $\{v'\} \cup N_{G'}(v')$ should have got decreased by the removal of Q from G. Let P be the set of vertices in $\{v'\} \cup N_{G'}(v')$ whose degrees got reduced due to the removal of Q from G, i.e., $P = \{z \in \{v'\} \cup N_{G'}(v') : deg_{G'}(z) < deg_G(z)\}$.

For a vertex $x \in V(G)$, let $M''_G(x) = \{u \in N_G(x) : deg_G(u) > 3\}$ and $M'_G(x) = N_G(x) - M''_G(x)$. Note that in all the configurations defined in $Lemma$ 7.4, the main criteria which characterizes each configuration is the degree of the pivot v' and the degrees of the vertices in $N'(v')$. We make the following claim:

Claim 7.30. *There exists a vertex x in P such that $|M''_G(x)| \leq 3$.*

Proof: It is easy to see that $M''_G(x) \subseteq N_{G'}(x)$. If there exists a vertex in P, whose degree is at most 3, say x, then we have $|M''_G(x)| \leq 3$. Thus we can assume that the degree of any vertex in P is at least 4.

Now suppose the pivot vertex v' is in P. Then let $x = v'$. It is clear that v' has to be in one of the configuration $B3 - B5$. In any of these configurations there can be at most two neighbours with degree greater than 3. Note that in this case all the degree 3 neighbours of $x = v'$ in G' are of degree 3 in G also since otherwise P will contain a vertex of degree at most 3, a contradiction. Thus we have $|M''_G(x)| \leq 2$.

The only remaining case is when $v' \notin P$. Since the degree of v' has not changed and $\{v'\} \cup N_G(v')$ was not in any configuration in G, it means that one of the vertex in $N'(v')$ has had its degree decreased. We call that vertex as x. Since the degree of any vertex in P is at least 4, $deg_{G'}(x) \geq 4$. Since we can have degree ≥ 4 vertex in $N'(v')$ only if $\{v'\} \cup N_G(v')$ forms a configuration $B2$, we infer that $deg_{G'}(x) = 4$. Moreover $deg_{G'}(v') = deg_G(v') = 3$. Thus we have $|M''_G(x)| \leq |N_{G'}(x) - \{v'\}| \leq 4 - 1 = 3$.

Thus we have $|M''_G(x)| \leq 3$.

□

In G, let y be a two degree neighbour of vertex x - selected in $Claim$ 7.30 - such that $N(y) = \{x, y'\}$. Now by induction $G - \{xy\}$ is acyclically edge colorable using $\Delta + 3$ colors. Let c' be a valid coloring of $G - \{xy\}$. With respect to the coloring c' let $F'_x(c') = \{c'(x, z) | z \in M'(x)\}$ and $F''_x(c') = \{c'(x, z) | z \in M''(x)\}$ i.e., $F''_x = F_x - F'_x$.

Now if $c'(y, y') \notin F_x$ we are done as there are at least three candidate colors which are also valid by $Lemma$ 2.3. We know by $Claim$ 7.30 that $|F''_x| \leq 3$. If $c'(y, y') \in F'_x$, then let $c = c'$. Else if $c'(y, y') \in F''_x$, then recolor edge yy' using a color from $C - (S_{yy'} \cup F''_x)$ to get a coloring c (Note that $|C - (S_{yy'} \cup F''_x)| \geq \Delta + 3 - (\Delta - 1 + 3) = 1$ and since y is a pendant vertex in $G - \{xy\}$ the recoloring is valid). Now if $c(y, y') \notin F_x$ the proof is already discussed. Thus $c(y, y') \in F'_x$.

With respect to coloring c, let $a \in M'(x)$ be such that $c(x, a) = c(y, y') = 1$. Now if none of the candidate colors in $C - (F_x \cup F_y)$ are valid for the edge xy, then all those candidate colors have to be actively present in S_{xa}, implying that $|S_{xa}| \geq |C - (F_x \cup F_y)| \geq \Delta + 3 - (\Delta - 1 + 1 - 1) = 4$, a contradiction since $|S_{xa}| \leq 2$ (Recall that $a \in M'(x)$ and $deg(a) \leq 3$). Thus we have a valid color for the edge xy, a contradiction to the assumption that G is a counter example.

■

Chapter 8

Lower Bounds and Dense Graphs

In this chapter lets look at the lower bounds for $a'(G)$. We also give exact bound for $K_{p,p}$.

8.1 Previous Results

By Vizing's theorem, we have $\Delta \leq \chi'(G) \leq \Delta+1$(see [18] for proof). Since any acyclic edge coloring is also proper, we have $a'(G) \geq \chi'(G) \geq \Delta$. There are graphs which require $\Delta+1$ colors to be properly colored. A natural questions that comes to mind is to ask if this bound ($\Delta+1$) the best possible lower bound for acyclic edge chromatic number also. We will soon see that the bound could be slightly improved. We start with the following claim:

Claim 8.1. *If G is a d-$regular$ graph, then $a'(G) \geq d+1$.*
Proof: The proof is by contradiction. Suppose G can be acyclically colored using only d colors. Choose any two colors, say α and β. Now start from a vertex with an edge colored α and trace the (α, β) bichromatic path. Now since both the colors α and β are available at each vertex and the graph is finite, the bichromatic path should return to the starting vertex thus completing a bichromatic cycle. This is a contradiction to the fact that G was acyclically edge colored using d colors. $\square$

By Claim 8.1, we get infinite classes of graphs which require at least $\Delta+1$ colors. Can we better this? The following claim does that:

Claim 8.2. $a'(K_{2n}) \geq 2n+1 = \Delta+2$.

Proof: The proof is by contradiction. Suppose K_{2n} can be acyclically colored using only $2n$ colors. We know that any colors can cover only n edges (perfect matching). There cannot be two color classes with n edges each since they would induce a bichromatic cycle. Thus any other color class than the first one can have only $n-1$ edges. Now if we cunt the number of edges covered, it will be $n+(2n-1)(n-1) = n(2n-1)-(n-1)$. But the number of edges in $K_{2n} = n(2n-1)$. Thus there are at least $(n-1)$ edges which does not belong to any color class, a contradiction. □

If we observe the proof carefully, we will see that even the last color class has $(n-1)$ edges. This means that even if we delete any $(n-2)$ edges from K_{2n}, it would require $2n+1$ colors to be acyclically edge colored.

Alon, Sudakov and Zaks [7] conjectured that complete graphs of even order are the only regular graphs which require $\Delta+2$ colors to be acyclically edge colored. Nešetřil and Wormald [36] supported the statement by showing that the acyclic edge chromatic number of a random d-regular graph is asymptotically almost surely equal to $d+1$ (when $d \geq 2$). In this chapter we show that this is not true in general.

8.2 Theorems on Dense Regular Graphs

Theorem 8.3. *Let G be a d-regular graph with $2n$ vertices and $d > n$, then $a'(G) \geq d+2 = \Delta(G)+2$.*

Proof: Observe that two different color classes cannot have n edges each, since that will lead to a bichromatic cycle. Therefore at most one color class can have n edges while all other color classes can have at most $n-1$ edges. Thus the number of edges in the union of $\Delta(G)+1 = d+1$ color classes is at most $n+d(n-1) < dn$, when $d > n$ (Note that dn is the total number of edges in G). Thus G needs at least one more color. Thus $a'(G) \geq d+2 = \Delta(G)+2$.

■

Remark: It is clear from the proof that if $n+d(n-1)+x < dn$ then even after removing x edges from the given graph, the resulting graph still would require $d+2$ colors to be acyclically edge colored.

Theorem 8.4. *For any d and n such that dn is even and $d \geq 5, n \geq 2d+3$, there exists a connected d-regular graphs that requires $d+2$ colors to be acyclically edge colored.*

Proof: If d is odd, let $G' = K_{d+1}$. Else if d is even let G' be the complement of a perfect matching on $d+2$ vertices. Let H be any d-regular graph on $N = n - n'$ vertices. Now remove an edge (a, a') from G' and an edge (b, b') from H. Now connect a to b and a' to b' to create a d-regular graph G. Clearly G requires $d+2$ colors to be acyclically edge colored since otherwise it would mean that $G' - \{(a, a')\}$ is $d+1$ colorable, a contradiction in view of the Remark following Theorem 1, for $d \geq 5$.

■

8.3 Complete Bipartite Graphs

8.3.1 Lower Bound for Complete Bipartite Graphs

Complete bipartite graphs offer a interesting case since they have $d = n$. Observe that the counting argument in Theorem 8.3 fails. We deal with this case in this section. Before going to the Theorem, let us look at a Lemma which helps us in the proof.

Lemma 8.5. *If n is even, then $K_{n,n}$ does not contain three disjoint perfect matchings M_1, M_2, M_3 such that $M_i \cup M_j$ forms a hamiltonian cycle for $i, j \in \{1, 2, 3\}$ and $i \neq j$.*

Proof: Observe that a perfect matching of $K_{n,n}$ corresponds to a permutation of $\{1, 2, \ldots, n\}$. Let the perfect matching M_i correspond to permutation π_i. Without loss of generality, we can assume that π_1 is the identity permutation by renumbering the vertices of one side of $K_{n,n}$.

Suppose $K_{n,n}$ contains three perfect matchings M_1, M_2, M_3 such that $M_i \cup M_j$ forms a hamiltonian cycle for $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Now we study the permutation $\pi_i^{-1}\pi_j$. Since $M_i \cup M_j$ induces a hamiltonian cycle in $K_{n,n}$, it is easy to see that the smallest $t \geq 1$ such that $(\pi_i^{-1}\pi_j)^t(1) = 1$ equals n. It follows that, in the cycle structure of $\pi_i^{-1}\pi_j$, there exists exactly one cycle and this cycle is of length n. The sign of a permutation is defined as: $sign(\pi) = (-1)^k$, where k is the number of even cycles in the cycle structure of the permutation π. Recalling that n is even, we have observe the following:

Observation 8.6. $sign(\pi_i^{-1}\pi_j) = -1$ *for* $i, j \in \{1, 2, 3\}$ *and* $i \neq j$.

Now with respect to $\pi_i^{-1}\pi_j$, taking $\pi_i = \pi_1$ (the identity permutation) and $\pi_j = \pi_2$ (or π_3), we infer that $sign(\pi_2) = -1$ and $sign(\pi_3) = -1$. Now $sign(\pi_2^{-1}\pi_3) = sign(\pi_2^{-1})sign(\pi_3)$ = (-1)(-1) = 1, a contradiction in view of $Observation$ 8.6.

■

Now we obtain the lower bound for complete bipartite graphs ($K_{n,n}$, when n is odd) in the following theorem:

Theorem 8.7. $a'(K_{n,n}) \geq n + 2 = \Delta + 2$, *when n is odd.*

Proof: Since $K_{n,n}$ is a regular graph, $a'(K_{n,n}) \geq \Delta + 1 = n + 1$. Suppose $n + 1$ colors are sufficient. This can be achieved only in the following way: One color class contains n edges and the remaining color classes contain $n - 1$ edges each. Let α be the color class that has n edges. Thus color α is present at every vertex on each side A and B. Any other color is missing at exactly one vertex on each side.

Observation 8.8. *Let* $\theta \neq \alpha$ *be a color class. The subgraph induced by color classes* θ *and* α *contains* $2n - 1$ *edges and since there are no bichromatic cycles, the subgraph induced is a hamiltonian path. We call this an* (α, θ) *hamiltonian path.*

Observation 8.9. *Let* θ_1 *and* θ_2 *be color classes with* $n - 1$ *edges each. The subgraph induced by color classes* θ_1 *and* θ_2 *contains* $2n - 2$ *edges. Since there are no bichromatic cycles, the subgraph induced consists of exactly two paths.*

Note that there is a unique color missing at each vertex on each side of $K_{n,n}$. Let $m(u)$ be the color missing at vertex u. For $a_1 \in A$ and $b_1 \in B$, let $m(a_1) = m(b_1) = \beta$. Let the color of the edge $(a_1, b_1) = \gamma$. Clearly $\gamma \neq \alpha$ since otherwise there cannot be a (α, β) hamiltonian path, a contradiction to $Observation$ 8.8. For $a_2 \in A$ and $b_2 \in B$, let $m(a_2) = m(b_2) = \gamma$. Its clear that $a_1 \neq a_2$ and $b_1 \neq b_2$. Consider the subgraph induced by the colors β and γ. In view of $Observation$ 8.9 it consists of exactly two paths. One of them is the single edge (a_1, b_1). The other path has length $2n - 3$ and has a_2 and b_2 as end points.

Now we construct a $K_{n+1,n+1}$ from the above $K_{n,n}$ by adding a new vertex, a_{n+1} to side A and a new vertex, b_{n+1} to side B. Now for $u \in B$ color each edge (a_{n+1}, u) by the color $m(u)$ and for $v \in A$ color each edge (b_{n+1}, v) by the color $m(v)$. Assign the color α to the edge (a_{n+1}, b_{n+1}). Clearly the coloring thus obtained is a proper coloring.

Now we know that there existed a (α, β) hamiltonian path in $K_{n,n}$ with a_1 and b_1 as end points. Recalling that $m(a_1) = m(b_1) = \beta$, we have $color(a_{n+1}, b_1) = color(b_{n+1}, a_1) = \beta$. It is easy to see that in $K_{n+1,n+1}$ this path along with the edges (a_1, b_{n+1}), (b_{n+1}, a_{n+1}) and (a_{n+1}, b_1) forms a (α, β) hamiltonian cycle. In a similar way, for (α, γ) hamiltonian path that existed in $K_{n,n}$, we can see that in $K_{n+1,n+1}$, we have a corresponding (α, γ) hamiltonian cycle.

Recall that there was a (β, γ) bichromatic path starting from a_2 and ending at b_2 in $K_{n,n}$. In the $K_{n+1,n+1}$ we created, we have $c(a_2, a_{n+1}) = \gamma$, $c(a_1, b_{n+1}) = \beta$, $c(a_{n+1}, b_1) = \beta$ and $c(a_{n+1}, b_2) = \gamma$. Thus the above (β, γ) bichromatic path in $K_{n,n}$ along with the edges (a_2, b_{n+1}), (b_{n+1}, a_1), (a_1, b_1), (b_1, a_{n+1}), (a_{n+1}, b_2) in that order forms a (β, γ) bichromatic hamiltonian cycle. Thus we have 3 perfect matchings induced by the color classes α, β and γ whose pairwise union gives rise to hamiltonian cycles in $K_{n+1,n+1}$, a contradiction to *Lemma* 8.5 since $n + 1$ is even.

■

8.3.2 Exact Bound for $K_{p,p}$

Theorem 8.10. $a'(K_{p,p}) \leq p + 2 = \Delta + 2$*, when* p *is an odd prime. By Theorem 8.7, this implies that* $a'(K_{p,p}) = p + 2 = \Delta + 2$.

Proof: Let $A = \{0, 1, \ldots, p-1\}$ and $B = \{0, 1, \ldots, p-1\}$. Let $\pi_0, \pi_1, \ldots, \pi_{p-1}$ be the permutation defined by $\pi_i : a \longmapsto (a + i) \pmod p$. Let M_i be the perfect matching corresponding to the permutation π_i. It is easy to verify that if $i \neq j$, then $M_i \cap M_j = \emptyset$. Now we claim the following:

Claim 8.11. *If* $i \neq j$*, then* $M_i \cup M_j$ *forms a Hamiltonian cycle (i.e.,* $M_0, M_1, \ldots M_{p-1}$ *form perfect 1-factorization).*

Proof: First note that the union of any two perfect matchings forms a collection of disjoint cycles. Suppose two matchings M_i and M_j, $(i > j)$ are such that a cycle of length $2k < 2p$ gets formed by the edges of $M_i \cup M_j$ (Recall that all cycles are of even length in $K_{p,p}$). Without loss of generality let this cycle contain the vertex $a \in A$. It is easy to see that $(\pi_j^{-1}\pi_i)^k(a) = a$. Noting that $(\pi_j^{-1}\pi_i)(a) = a + i - j \pmod p$, we have $(\pi_j^{-1}\pi_i)^k(a) = a + ki - kj = a + k(i - j) \pmod p = a \pmod p$, which implies that $k(i - j) = 0 \pmod p$. Since $i - j \neq 0$

$\pmod p$, we have $k = 0 \pmod p$, a contradiction since $k < p$. Thus $M_i \cup M_j$ forms a cycle of length $2p$ (a Hamiltonian cycle) when i and j are distinct. $\square$

Now consider the multiplicative group Z_p^*, and let x be a generator of this group. Define a permutation π of $\{1, 2, \ldots, p-1\}$ by $\pi : a \longmapsto ax \pmod p$. Let M be the matching corresponding to the permutation π.

Claim 8.12. $|M \cap M_i| = 1$ *, for each* M_i*,* $1 \leq i \leq p-1$ *and* $M_0 \cap M = \emptyset$ *(i.e., for each* M_i*,* $1 \leq i \leq p-1$*, The matchings* M *and* M_i *exactly have one edge in common. Also the matchings* M *and* M_0 *do not have any edge in common).*

Proof: By the definition of M, we infer that $M_0 \cap M = \emptyset$. Now let $a = i(x-1)^{-1} \pmod p$. Note that since $i \neq 0$, $a \neq 0$. We have $\pi_i(a) = a + i = i(x-1)^{-1} + i = i(x-1)^{-1}(1 + x - 1) = i(x-1)^{-1}(x) = ax \pmod p = \pi(a)$. Thus it follows that the edge $(a, ax) \in M \cap M_i$ for $a = i(x-1)^{-1} \pmod p$. Therefore $|M \cap M_i| \geq 1$ for $1 \leq i \leq p-1$. Since $|M| = p-1$, we can also infer that $|M \cap M_i| = 1$. $\square$

Now color the edges of $K_{p,p}$ as follows to get a coloring f with $p+2$ colors:

1. if $e \in M_i \setminus M$ (where $0 \leq i \leq p-1$), then it is colored with color c_i.
2. if $e \in M - (1, x)$, then it is colored with color c_p.
3. edge $e = (1, x)$ is colored with color c_{p+1}.

Claim 8.13. *The coloring* f *is acyclic.*

Proof: Obviously f is a proper coloring. Let c_i and c_j be two colors. We consider different values for i and j with $i > j$ and show that (c_i, c_j) bichromatic cycle cannot exist.

case 1: $i = p+1$

Since there is only one edge colored c_{p+1}, there cannot be any bichromatic cycle involving the color c_{p+1}.

case 2: $i, j < p$

Note that $M_i \cup M_j$ forms a Hamiltonian cycle by $Claim$ 8.11. Now at least one edge of M_i belongs to M (By $Claim$ 8.12) and is colored c_p or c_{p+1} with respect to the coloring f, breaking the possible (c_i, c_j) bichromatic cycle. Therefore there cannot be any (c_i, c_j) bichromatic cycle when $i, j < p$.

case 3: $i = p$

Suppose M_j is a matching such that a cycle of length $2k < 2p$ (no cycles of length $2p$ can be formed as there are only $p-2$ edges of color c_p) gets formed by the edges of $M \cup M_j$ (Recall that all cycles are of even length in $K_{p,p}$). Thus $(\pi_j^{-1}\pi)^k(a) = a \pmod{p}$. Noting that $(\pi_j^{-1}\pi)(a) = ax - j \pmod{p}$, we have $(\pi_j^{-1}\pi)^2(a) = (ax-j)x - j = ax^2 - j(x+1) \pmod{p}$. Similarly $(\pi_j^{-1}\pi)^k(a) = ax^k - j(x^{k-1} + ... + x + 1)$ = $ax^k - j(x^k-1)(x-1)^{-1} = a \pmod{p}$. We have $a(x^k-1) - j(x^k-1)(x-1)^{-1} = 0 \pmod{p}$ and thus $(x^k-1)(a - j(x-1)^{-1}) = 0 \pmod{p}$. If $(a - j(x-1)^{-1}) = 0 \pmod{p}$, then $a = j(x-1)^{-1} \pmod{p}$. But according to $Claim$ 8.12, edge $(a, ax) \in M \cap M_j$. Therefore this edge and thus vertex a cannot be in the cycle formed by $M \cup M_j$, a contradiction. Thus we infer that $(x^k - 1) = 0 \pmod{p}$. This implies that $x^k = 1 \pmod{p}$ and hence $k = p-1$, since x is a generator. Thus there are $2(p-1)$ edges in the cycle, out of which $p-1$ are colored c_p, a contradiction since only $p-2$ edges are colored c_p. □

This completes the proof.

■

Notice that the last color was used for only one edge in the above coloring of $K_{p,p}$. Thus we get the following Theorem:

Theorem 8.14. *For a prime $p > 2$, if G is a graph obtained by removing just one edge from $K_{p,p}$, then $a'(G) = \Delta + 1 = p + 1$ (The above statement is true even if we delete any number of edges between 1 and $p-2$).*

Proof: It is easy to infer from the proof of Theorem 8.10 that $a'(G) \le p+1$. The lower bound comes from a simple counting argument: At most one color class can have p edges, since otherwise there will be bichromatic cycles. Thus if $a'(G) \le p$, then there can be at most $p + (p-1)^2 < p^2 - 1$ edges in G, a contradiction.

■

8.4 Remarks

1. It is interesting to compare the statement of Theorem 1 to the result of [36], namely that almost all d-regular graphs for a fixed d, require only $d+1$ colors to be acyclically edge colored. From the introduction of [36], it appears that the authors expect their result for random d-regular graphs would extend to all d-regular graphs except for K_n, n even. From Theorem 8.3 and Theorem 8.4 it is clear that this is not true: There exists a large number of d-regular graphs which require $d+2$ colors to be acyclically edge colored, even when d is fixed.

2. The complete bipartite graph, $K_{n+2,n+2}$ is said to have a perfect 1-factorization if the edges of $K_{n+2,n+2}$ can be decomposed into $n+2$ disjoint perfect matchings such that the union of any two perfect matchings forms a hamiltonian cycle. It is obvious from Lemma 8.5 that $K_{n+2,n+2}$ does not have perfect 1-factorization when n is even. When n is odd, some families have been proved to have perfect 1-factorization (see [15] for further details). It is easy to see that if $K_{n+2,n+2}$ has a perfect 1-factorization then $K_{n+2,n+1}$ and therefore $K_{n+1,n+1}$ has a acyclic edge coloring using $n+2$ colors. Therefore the statement of Theorem 8.7 cannot be extended to the case when n is even in general.

3. Clearly if $K_{n+2,n+2}$ has a perfect 1-factorization, then $a'(K_{n,n}) = n+2$. It is known that (see [15]), if $n+2 \in \{p, 2p-1, p^2\}$, where p is an odd prime or when $n+2 < 50$ and odd, then $K_{n+2,n+2}$ has a perfect 1-factorization. Thus the lower bound in Theorem 8.7 is tight for the above mentioned values of $n+2$. As of now, these are the only values of n for which we know the exact value for $a'(K_{n,n})$. Note that we cannot apply the simple argument mentioned here when $n = p$. Alon, McDiarmid and Reed [6] observed that $a'(K_{p-1,p-1}) = p$.

4. To get an upper bound for $a'(K_{n,n})$, the best method we can think of is to look for the smallest prime number p such that $p \geq n+2$. Then $a'(K_{n,n}) \leq p$. A weakening of the result of Iwaniec and Pintz [30] gives that for every sufficiently large integer x, there exists a prime number in the range $[x, x + x^{0.6}]$.

Chapter 9

Conclusion

9.1 Open Problems

9.1.1 Acyclic Edge Coloring Conjecture

The conjecture is still open. The conjecture has been proved for some special classes of graphs. This indicates that the conjecture might be true. One can attempt the conjecture. Till now the best bound of 16Δ uses probabilistic methods to obtain the bound. Such methods may not work to get the conjectured bound of $\Delta + 2$. Constructive methods need to be developed. But till now such methods have only been used for sparse graphs. The global nature of the problem has to be investigated throughly to come up with concepts which would lead to closer upper bounds towards the conjectured bound.

Another direction is to try to disprove the conjecture by constructing counter examples. Our results on regular graphs which require $\Delta + 2$ colors are the first steps towards such an approach. The results on lower bounds use counting arguments. But such techniques might not work in the attempt to disprove the conjecture. Clever techniques other than counting are needed to be explored to find better lower bounds.

Planar Graphs

The best known bound for planar graphs as of now is $\Delta + 12$. We feel that this bound could be improved. It would be interesting to prove the conjecture for planar graphs. Even for triangle free planar graphs, the conjecture is still open. Another question is to investigate examples of planar graph other than K_4 which requires greater than $\Delta + 1$ colors.

9.1.2 Complete Graphs

Is it true that $a'(K_{2n+1}) = 2n + 1$, $\forall n \geq 1$? This problem of determining the acyclic chromatic index for complete graphs takes importance due to its equivalence with the perfect 1-factorization conjecture.

9.1.3 Algorithmic Questions

The research till now has mostly concentrated towards finding better bounds on acyclic chromatic index. Since most of the results on special classes of graphs use constructive methods, there exists an underlying polynomial time algorithm. But very few results look at time complexity in a serious manner. Thus various algorithmic questions are open to be explored.

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