Randomized Algorithms For Finding Matchings In Bipartite And General Graphs

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Outline of The Presentation

- Skew-Symmetric Matrices and Matchings
- Lovász's Theorem
- Rabin - Vazirani's Algorithm
- Matchings via Gaussian Elimination

Skew-Symmetric Matrices and Matchings

Tutte Matrix

- $$\begin{aligned} A_{i,j} &= x_{i,j} && \text{if } (v_i, v_j) \in E \text{ and } i < j. \\ &= -x_{i,j} && \text{if } (v_i, v_j) \in E \text{ and } i > j \\ &= 0 && \text{otherwise} \end{aligned}$$
- Partition $P = \{\{i_1, j_1\}, \cdots, \{i_n, j_n\}\}$.
- Define Pfaffian of A as
  pf$A = \sum_P sgn(P).b_{i_1 j_1} \cdots b_{i_n j_n}$
- det$A = (\text{pf}B)^2$
- Tutte's Theorem:
  $|A| \neq 0 \Leftrightarrow G$ has a perfect matching

Lovász's Theorem

Frobenius's Theorem

- Theorem(Lovász):
  Let q be the size of a maximum matching in G.
  Then rank(A)=$2q$.
- Frobenius's Theorem:
  Let $\alpha, \beta \subseteq \{1, 2, \cdots, n\}, |\alpha| = |\beta| = \text{rank}(A)$.
  Then,
  $|A_{\alpha\alpha}|.|A_{\beta\beta}| = (-1)^{|\alpha|}|A_{\alpha\beta}|^2$

Proof(Lovász's Theorem):

1 Rank(A) $\geq 2.q$:
Let U be vertices matched by a matching of size q.
$G(U)$ has a perfect matching. So, $|A_{UU}| \neq 0$.
$|U| = 2.q \Rightarrow$ rank(A)$\geq 2.q$.

Proof(Lovász's Theorem):

1. Rank(A) $\geq 2.q$:
   Let U be vertices matched by a matching of size q.
   $G(U)$ has a perfect matching. So, $|A_{UU}| \neq 0$.
   $|U| = 2.q \Rightarrow$ rank(A)$\geq 2.q$.
2. Rank(A) $\leq 2.q$:
   Let Rank(A)=k. Let $|A_{\alpha\beta} \neq 0|, |\alpha| = |\beta| = k$
   By Frobenius's Theorem, $|A_{\alpha\alpha} \neq 0$.
   So, by Tutte's Theorem, $G(\alpha)$ must have a perfect matching.
   So, G has a matching of size $k/2$.

Some Lemmas on Tutte Matrix of G

Lemma 1: If $|A| \neq 0$, for any prime p, $|A| \bmod p \neq 0$.
Proof:
Let $M = \{(i_1, j_1), \cdots, (i_q, j_q)\}$ be a matching.
Now, $|A|$ contains the monomial $x_{i_1 j_1}^2 \cdots x_{i_q j_q}^2$
with co-efficient 1.
So, $|A| \bmod p \neq 0$.

Some Lemmas on Tutte Matrix of G

Lemma 1: If $|A| \neq 0$, for any prime p, $|A|$ mod $p \neq 0$.
Proof:
Let $M = \{(i_1, j_1), \cdots, (i_q, j_q)\}$ be a matching.
Now, $|A|$ contains the monomial $x^2_{i_1 j_1} \cdots x^2_{i_q j_q}$
with co-efficient 1.
So, $|A|$ mod $p \neq 0$.

Lemma 2: If rank$(A) = k$, then for any prime p, there exists a $k \times k$ submatrix C such that $|C| \neq 0$ and $|C| mod p \neq 0$.
Proof:
Since rank$(A) = k$, there exists a $k \times k$ submatrix $A_{\alpha\beta}$
such that $|A_{\alpha\beta}| \neq 0$.
By Fobenius's Theorem, $|A_{\alpha\alpha}| \neq 0$.

Lemma 3: Let A be of even dimension with entires in a field F.
For $i \neq j$, if $|A_{ij}| \neq 0$, then $|A_{ii,jj}| \neq 0$.
Proof:
Take i =1, j =2.
$|A_{12}| \neq 0 \Rightarrow n-1$ columns of A_{12} are linearly independent.
Particularly, columns numbered by $\beta = \{3, \cdots, n\}$ are linearly independent.
For some $\alpha \subset \{2, \cdots, n\}, |\alpha| = n-2$, $|A_{\alpha\beta}| \neq 0$.
$|A_{11}| = 0$. Since, $|A_{\alpha\beta}| \neq 0$, rank(A_{11})= $n-2$.
By Frobenius's theorem, $|A_{\beta\beta}| = |A_{11,22}| \neq 0$.

Rabin-Vazirani's Algorithm

Rabin - Vazirani's Algorithm

Claim: Let S be a good substitution, and $B=(A^S)^{-1}$.
$\exists\ j$ such that $S(a_{ij}) \neq 0$ mod p and $b_{j1} \neq 0$ mod p.
Proof:
$|A^S| = \sum_{j=1}^{n} (-1)^{1+j} S(a_{1j})|A^S_{1j}| \neq 0$.
$\exists j$ such that $S(a_{ij}) \neq 0$ mod p and $|A^S_{1j}| \neq 0$ mod p.
But, $b_{j1} \neq 0$ mod p.
By Lemma 3, $|A^S_{11,jj}| \neq 0$ mod p.

- Add $n - 2q$ new vertices.
- Using Gallai-Edmonds structure theorem, we get an upper bound on the size of maximum matching in G.
  Run these two Monte-Carlo Algorithms till they agree.

Using Gaussian Elimination

Running Time

Cost of update in i-th iteration is proportional to cost of multiplying a $2^j \times 2^j$ matrix by a $2^j \times n$ matrix.
This can be done in time $n \cdot (2^j)^{\omega-1}$.
But every j appears $n/2^j$ times.

References

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[3] L. Lovász and M. Plummer. Matching Theory. Akadémiai Kiadó, 1986.