Boxicity, Cubicity and Vertex Cover

A Thesis
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by

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Abstract

The boxicity of a graph G, denoted as $box(G)$, is the minimum dimension d for which each vertex of G can be mapped to a d-dimensional axis-parallel box in $\mathbb{R}^d$ such that two boxes intersect if and only if the corresponding vertices of G are adjacent. An axis-parallel box is a generalized rectangle with sides parallel to the coordinate axes. If additionally, we restrict all sides of the rectangle to be of unit length, the new parameter so obtained is called the cubicity of the graph G, denoted by $cub(G)$.

F.S. Roberts had shown that for a graph G with n vertices, $box(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $cub(G) \leq \left\lfloor \frac{2}{3}n \right\rfloor$. A minimum vertex cover of a graph G is a minimum cardinality subset S of the vertex set of G such that each edge of G has at least one endpoint in S. We show that $box(G) \leq \left\lfloor \frac{t}{2} \right\rfloor + 1$ and $cub(G) \leq t + \lceil log_2(n-t) \rceil - 1$ where t is the cardinality of a minimum vertex cover. Both these bounds are tight.

For a bipartite graph G, we show that $box(G) \leq \left\lceil \frac{n}{4} \right\rceil$ and this bound is tight. We observe that there exist graphs of very high boxicity but with very low chromatic number. For example, there exist bipartite (2 colorable) graphs with boxicity equal to $\frac{n}{4}$. Interestingly, if boxicity is very close to $\frac{n}{2}$, then the chromatic number also has to be very high. In particular, we show that if $box(G) = \frac{n}{2} - s$, $s \geq 0$, then $\chi(G) \geq \frac{n}{2s+2}$, where $\chi(G)$ is the chromatic number of G.

We also discuss some known techniques for finding an upper bound on the boxicity of a graph - representing the graph as the intersection of graphs with boxicity 1 (boxicity 1 graphs are known as interval graphs) and covering the complement of the graph by co-interval graphs (a co-interval graph is the complement of an interval graph).

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Chapter 1

Introduction

In this thesis we discuss some dimensional properties of graphs - boxicity and cubicity. This thesis is not meant to be a comprehensive survey on these dimensions, but rather we start by introducing these concepts so that the reader becomes familiar with them and then discuss some known properties which we shall use later while giving new bounds on these dimensions. Some basic knowledge of graph theory is assumed. Otherwise, the thesis is mostly self-contained.

The basic assumptions and notations used in this thesis are discussed in Section 1.1. In Section 1.2 we discuss how the problem of finding the boxicity of a graph originated in ecology. We give a brief overview of the thesis in Sections 1.3, 1.4 and 1.5 and also state our contributions. In these sections we won't be giving any definitions, which are deferred till the relevant chapters. We discuss the organization of the other chapters of this thesis in Section 1.6, so that the reader familiar with the notion of boxicity and cubicity can read the topics of interest to him/her self.

1.1 Basics and Notation

Throughout this thesis, a graph should be understood as a simple, finite, undirected and connected graph, unless otherwise stated. We use the notation, terminology and definitions used in [12]. We shall denote the number of vertices of a graph by $|G|$ or

n and the number of edges by $||G||$ or m. An edge with ends a and b is considered as a 2-tuple (a, b). Graphs are considred equal under isomorphism. For a graph G, let $N_G(v) = \{u \in V(G)|(u, v) \in E(G)\}$ denote the set of neighbours of a vertex $v \in V(G)$. P^n denotes the path on $n + 1$ vertices and C^n denotes the cycle on n vertices. If $G = (V, E)$ and $H = (V, F)$, then $G \cup H = (V, E \cup F)$ and $G \cap H = (V, E \cap F)$. A bipartite graph $G = (V_1 \cup V_2, E)$ is used to denote a bipartite graph with a bipartition $\{V_1, V_2\}$ of the vertex set such that V_1 and V_2 are independent sets. The Cartesian graph product $G_1 \Box G_2$, also called the graph product of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph G with vertex set $V_1 \times V_2$ and for $u = (u_1, u_2) \in V_1 \times V_2$ and $v = (v_1, v_2) \in V_1 \times V_2$, $(u, v) \in E(G)$ if and only if either:
a) $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or
b) $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.
Other definitions and notations will be discussed in the relevant Chapters and Sections.

1.2 Food Webs, Interval Graphs and Boxicity

In ecology, *niche* is a term describing the relational position of a species or population in its ecosystem. The different dimensions of a niche represent different biotic and abiotic factors like geographic range, temperature range, habitat, place in the food chain, moisture, degree of acidity (pH range), amount of nutrients, time of the day, etc.

If we restrict ourselves to l independent factors, then the ecological niche these factors define occupies a subspace of the l-dimensional Euclidean space (ecological phase space). Two species compete if their ecological niches have non-empty intersection. It is difficult to know all the factors which determine an ecological niche, and some factors may be relatively unimportant and dependent on each other. Hence it is useful to start with the concept of competition and try to represent competition by niche overlap such that each ecological niche occupies a d dimensional subspace of a d-dimensional phase space.

It seems desirable that in an ideal representation, the factors determining the dimension of the ecological phase space would be independent and the niches would be

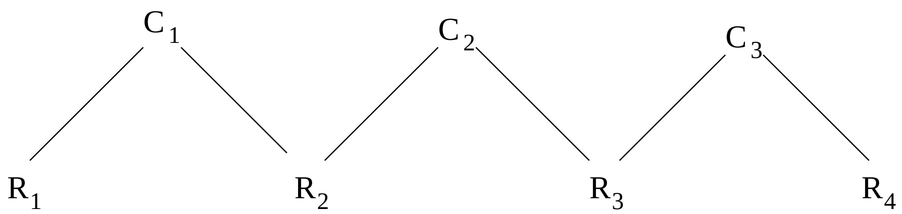

Figure 1.1: A simple foodweb

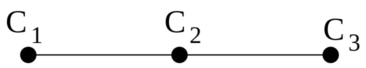

Figure 1.2: The graph derived from the food web

represented as axes-parallel boxes, or Cartesian products of intervals [17]. This gives rise to the following question:

> What is the minimum dimension of a niche space necessary to represent the overlaps among observed niches?

An *interval graph* is a well known class of graphs for which each vertex of the graph can be mapped to a closed interval on the real line (1 dimensional Euclidean space) in such a way that two intervals overlap if and only if the corresponding vertices are adjacent in the graph. For more information on interval graphs, read Chapter 2.2.

Joel E. Cohen [7, 8] used interval graphs to study the dimensionality of ecological niche space. Suppose we have a food web comprising of three consumers and four resources as shown in figure 1.1 [27]. If we consider the consumers to be vertices of a graph and make two vertices adjacent if and only if the corresponding consumers compete for at least one resource, then we have derived a graph from the food web as shown in figure 1.2. This graph is an interval graph with an interval representation as shown in figure 1.3.

The reader is encouraged to verify for him/her self that the only way in which we

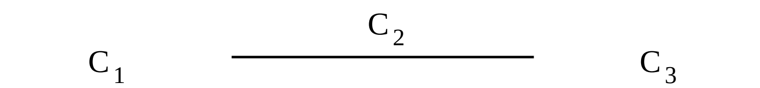

Figure 1.3: An interval representation of the derived graph

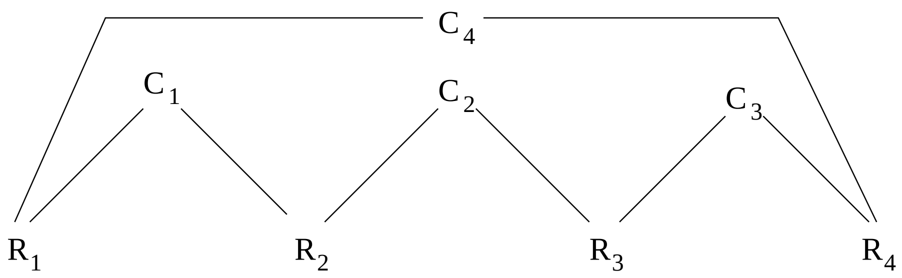

Figure 1.4: A foodweb whose derived graph is not an interval graph

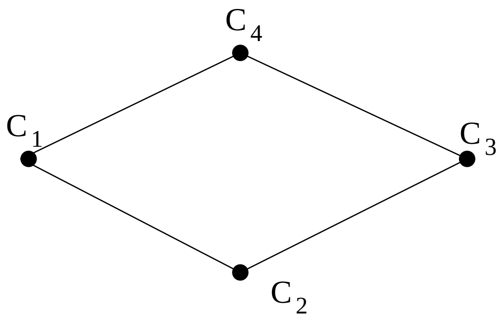

Figure 1.5: Graph derived by adding the fourth consumer

can add a fourth consumer to our food web so that the graph derived from the new food web doesn't correspond to an interval graph is as shown in figure 1.4. So, we are saying that the cycle of length 4 shown in figure 1.5 (the graph derived from the foodweb in figure 1.4) is not an interval graph. See Lemma 2.2 for a proof. So, for the example in figure 1.1, only 1 dimension is required to represent the ecological phase whereas for the example in figure 1.4, more than 1 dimension is required; but 2 dimensions are enough as demonstrated by the derived graph and its representation in 2 dimensions (figures 1.5 and 1.6).

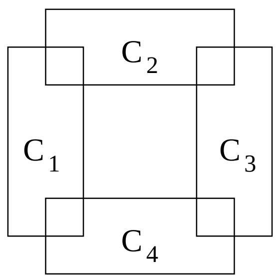

Figure 1.6: A 2-dimensional representation of the derived graph in figure 1.5

The minimum number of dimensions required to represent the ecological niche (or the graph derived from the food web) is known as its *boxicity*. Our example of figure 1.2 has boxicity 1 and the example in figure 1.5 has boxicity 2.

1.3 Boxicity and Cubicity of Graphs

F.S. Roberts studied the problem mentioned in Section 1.2 [20, 21] and formulated it mathematically as a generalization of interval graphs to higher dimensions [22]. He also introduced the notion of cubicity [22].

The *cubicity* of a graph G is the minimum dimension d for which each vertex of G can be mapped to an axis-parallel d-dimensional unit cube in $\mathbb{R}^d$ such that two cubes intersect if and only if the corresponding vertices of G are adjacent.

The boxicity and cubicity of a graph G are denoted by $box(G)$ and $cub(G)$ respectively throughout the thesis. Roberts [22] had shown that $box(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $cub(G) \leq \left\lfloor \frac{2}{3}n \right\rfloor$ for any graph G.

1.4 Boxicity, Cubicity and Minimum Vertex Cover

A vertex cover is a subset S of the vertex set of G such that each edge of G has at least one endpoint in S. In other words, for each edge (a, b) in E, at least one of a or b must be an element of S. A minimum vertex cover, denoted by MVC, is a vertex cover having the smallest cardinality among all the vertex covers. The minimum vertex cover need not be unique, but all such vertex covers have the same cardinality t.

We prove that $box(G) \leq \left\lfloor \frac{t}{2} \right\rfloor + 1$ (Theorem 3.16) and $cub(G) \leq t + \lceil log_2(n-t) \rceil - 1$ (Theorem 4.3) for any graph G [2] where $t = |MVC|$. Both these bounds are tight.

1.5 Bipartite Graphs and Chromatic Number

For a bipartite graph $G = (V_1 \cup V_2, E)$, we show that $box(G) \leq min\left\{ \left\lceil \frac{|V_1|}{2} \right\rceil, \left\lceil \frac{|V_2|}{2} \right\rceil \right\}$ (Theorem 3.20) and $cub(G) \leq \frac{n}{2} + \lceil \log n \rceil - 1$ (Corollary 4.4). Since either $|V_1| \leq \frac{n}{2}$ or

$|V_2| \leq \frac{n}{2}$, $box(G) \leq \lceil \frac{n}{4} \rceil$. In Section 3.5.1 we give an example to show that this bound is tight.

Thus, as shown in the example mentioned above, there exist graphs with very high boxicity (equal to $\frac{n}{4}$) but very low chromatic number (equal to 2). However, in Theorem 3.21 we show that if $box(G) = \frac{n}{2} - s$, $s \geq 0$, then $\chi(G) \geq \frac{n}{2s+2}$, where $\chi(G)$ is the chromatic number of G [2]. In other words when the boxicity is very close to $\frac{n}{2}$, the chromatic number also has to be very high. Note that $s \geq 0$ since $box(G) \leq \frac{n}{2}$ as shown in [22], Theorem 3.11 and Corollary 3.17.

1.6 Organization of the Rest of the Thesis

In Chapter 2 we discuss intersection graphs and a general framework - Dimensional Properties of graphs - for boxicity and cubicity. In Chapter 3 we discuss some techniques for giving bounds on the boxicity of graphs and then give an upper bound on the boxicity of general graphs in terms of the cardinality of a minimum vertex cover. We also give an upper bound on the boxicity of bipartite graphs and see the relationship between boxicity and chromatic number. In Chapter 4 we give an upper bound on the cubicity of general graphs in terms of the cardinality of a minimum vertex cover.

Chapters 3 and 4 can be read independently of each other after reading Chapter 2.3 and Theorems 2.8 and 2.9 from Chapter 2.4.

Chapter 2

Dimensional Properties

2.1 Introduction

In Section 2.2, we discuss intersection graphs like interval graphs and unit interval graphs. In Section 2.3 we formally introduce the boxicity and cubicity of graphs. In Section 2.4, we see a general framework - dimensional properties of graphs - of which interval graphs and unit interval graphs are specific instances. We also see how boxicity and cubicity are related to interval graphs and unit interval graphs respectively. In Section 2.5, we discuss the boxicity and cubicity of some simple graph classes and also see the relationship between boxicity and cubicity.

2.2 Intersection Graphs

Let S be a set and $\mathcal{F} = \{S_1, S_2, \ldots, S_p\}$ be a family of sets such that $\bigcup_{i=1}^{p} S_i = S$.

Definition 2.1. *The intersection graph $\Omega(\mathcal{F})$ of $\mathcal{F}$ has $S_1, S_2, \ldots, S_p$ as its vertices and an edge between two vertices if and only if the corresponding sets have nonempty intersection.*

A graph G is an intersection graph on S if there exists a family $\mathcal{F}$ of subsets of S such that G and $\Omega(\mathcal{F})$ are isomorphic graphs. For example, if $S = \{1, 2, 3, 4\}$, then P^2 (the path on 3 vertices) is an intersection graph on S since for $\mathcal{F} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$,

$\Omega(\mathcal{F}) = P^2$.

Definition 2.2. *An interval graph is an intersection graph of a family of closed intervals in* $\mathbb{R}^1$ *(the real number line).*

In other words, in a representation $\mathcal{F}$ of an interval graph, each subset S_i is a set of continuous points in $\mathbb{R}^1$ (an interval). See [15] for more information on interval graphs. For an interval graph, we shall represent the interval corresponding to a vertex v (in a representation of the interval graph) as $[l(v), r(v)]$.

Lemma 2.1. *P^n is an interval graph, $n \geq 0$.*

Proof. Let $P^n = v_0, v_1, \ldots, v_n$ be a path. We associate with each vertex v_i the interval $[i, i+1]$. Note that since $r(v_i) = i+1 = l(v_{i+1})$, the interval associated with each vertex overlaps with the intervals associated with its neighbouring vertices (in P^n) and since $r(v_i) \leq l(v_j)$ for $j \geq i+2$, the interval associated with any vertex does not overlap with the intervals associated with any vertices other than its neighbours (in P^n). Thus we have shown a representation of P^n as the intersection graph of closed intervals in $\mathbb{R}^1$. □

Lemma 2.2. *C^n is not an interval graph, $n \geq 4$.*

Proof. Let $v_0 \ldots v_{n-1} v_0$ be the cycle C^n. Assume for the sake of contradiction that C^n can be represented as the intersection of closed intervals in $\mathbb{R}^1$. Let $l(v_0) = a$, $r(v_0) = b$, $l(v_2) = c$ and $r(v_2) = d$. Since $(v_0, v_2) \notin E(C^n)$, the corresponding intervals cannot overlap in any interval representation of C^n. Either $b < c$ or $d < a$. Without loss of generality, we may assume that $b < c$. Then, $(v_0, v_1) \in E(C^n) \Rightarrow l(v_1) \leq b$ and $(v_1, v_2) \in E(C^n) \Rightarrow r(v_1) \geq c$. Thus, $[b, c] \subseteq [l(v_1), r(v_1)]$ and $b \in [l(v_1), r(v_1)]$. Also, $(v_2, v_3) \in E(C^n) \Rightarrow r(v_3) \geq c$ and $(v_{n-1}, v_0) \in E(C^n) \Rightarrow l(v_{n-1}) \leq b$. Since $v_2 v_3 \ldots v_{n-1} v_0$ is an induced path of C^n, $b \in [l(v_k), r(v_k)]$ for some k, $3 \leq k \leq n-1$. Then, the intervals corresponding to v_1 and v_k overlap at b which is a contradiction since $(v_1, v_k) \notin E(C^n)$ for $3 \leq k \leq n-1$. □

Definition 2.3. *An indifference graph is an intersection graph of a family of unit length closed intervals on* $\mathbb{R}^1$.

Equivalently, we can say that a graph G is an indifference graph if there exists a mapping $f : V(G) \to \mathbb{R}$ such that $\forall\ a, b \in V(G),\ |f(a) - f(b)| \leq 1 \Leftrightarrow (a, b) \in E(G)$.

We shall interchangeably refer to a unit interval $[a - 0.5, a + 0.5]$ by a. We could also equivalently have defined an indifference graph as an intersection graph of a family of closed intervals of a constant length c. To see the equivalence, just scale the intervals in a unit interval representation of the indifference graph by a factor of c around the origin.

2.3 Boxicity and Cubicity

A d-dimensional box in the d-dimensional Euclidean space is a generalized d-dimensional rectangle with sides parallel to the coordinate axes. A unit d-dimensional cube in the d-dimensional Euclidean space is a d-dimensional cube (a d-dimensional box with all sides of equal length) with all sides of unit length and parallel to the coordinate axes. We shall represent the d-dimensional unit cube $[a_1 - 0.5, a_1 + 0.5] \times [a_2 - 0.5, a_2 + 0.5] \times \cdots \times [a_d - 0.5, a_d + 0.5]$ by $(a_1, a_2, \ldots, a_d)$. Two unit cubes $a = (a_1, a_2, \ldots, a_d)$ and $b = (b_1, b_2, \ldots, b_d)$ intersect each other if and only if $\rho(a, b) \leq 1$ where ρ is the product metric:

$$\rho((a_1, a_2, \ldots, a_d), (b_1, b_2, \ldots, b_d)) = \max_i |a_i - b_i|. \tag{2.1}$$

This is true since if $|a_j - b_j| > 1$, $1 \leq j \leq d$, then the two cubes don't intersect along the j-th axis. A representation of a graph G as the intersection graph of boxes (unit cubes) in $\mathbb{R}^d$ shall be called a box (unit cube) representation of G in d dimensions. We are interested in finding the minimum d' such that a graph G has a unit cube representation in d' dimensions and the minimum d'' such that G has a box representation in d'' dimensions. But this raises a very basic question as to whether every graph has a unit cube representation (or box representation) in d dimensions for some d.

LEMMA 2.3 ([22]). *Every graph has a unit cube representation in n dimensions.*

Proof. It is equivalent to say that G has a unit cube representation in n dimensions or that there is a mapping $f : V(G) \to \mathbb{R}^n$ such that

$$\rho(f(a), f(b)) \leq 1 \Leftrightarrow (a, b) \in E(G); \ \forall \ a, b \in V(G)$$

Let $f_i(a)$ denote the i-th element of of the n-tuple $f(a), 1 \leq i \leq n$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and for $u \in V(G)$, let

$$\begin{aligned} f_i(u) &= 0 \qquad if\ u = v_i \\ &= 1 \qquad if\ u \in N_G(v_i). \\ &= 2 \qquad if\ u \in V(G) - \{N_G(v_i) \cup \{v_i\}\}. \end{aligned}$$

For $x, y \in V(G)$ where $x = v_k, y = v_l$, there are 2 cases to consider:
Case 1: $(x, y) \in E(G)$.
$f_i(x) - f_i(y) \leq 1\ \forall i, 1 \leq i \leq n$ since $f_k(x) = 0, f_k(y) = 1, f_l(y) = 0, f_l(x) = 1$ and $\{f_i(x), f_i(y)\} \in \{1, 2\}\ \forall i \neq k, l$. So, $\rho(f(x), f(y)) \leq 1$.
Case 2: $(x, y) \notin E(G)$.
$\rho(f(x), f(y)) = 2$ since $f_k(x) = 0$ and $f_k(y) = 2$.

Thus $\rho(f(x), f(y)) \leq 1 \Leftrightarrow (x, y) \in E(G)$. □

Since a unit cube is also a box, every graph has a box representation in n dimensions.

Definition 2.4. *The boxicity of a graph G denoted as $box(G)$ is the minimum d such that G has a box representation in d dimensions.*

Definition 2.5. *The cubicity of a graph G denoted as $cub(G)$ is the minimum d such that G has a unit cube representation in d dimensions.*

By convention, $box(K^n) = 0 = cub(K^n)$.

Proposition 2.4. *$box(G) \leq cub(G)$.*

Proof. Since any unit cube is also a box, a unit cube representation of a graph is also a box representation for the graph. □

It is equivalent to consider a representation of a graph by closed intervals (boxes) or open intervals (boxes) [22].

Proposition 2.5. *If H is an induced subgraph of G, then $box(H) \leq box(G)$ and $cub(H) \leq cub(G)$.*

Proof. Consider a box representation of G in $box(G)$ dimensions. Mapping the vertices in H to the same boxes they are mapped to in this box representation of G gives us a box representation of H in $box(G)$ dimensions. Thus, $box(H) \leq box(G)$. Similarly, $cub(H) \leq cub(G)$. □

PROPOSITION 2.6. *If G has a box (unit cube) representation in d dimensions, then G has a box (unit cube) representation in $d+e$ dimensions $\forall e \geq 0$.*

Proof. Consider a box representation of G in d dimensions. In this box representation, let the box corresponding to vertex v_i be $[a_{i_1}, b_{i_1}] \times [a_{i_2}, b_{i_2}] \times \cdots \times [a_{i_d}, b_{i_d}]$. Now, mapping vertex v_i to the $d+1$ dimensional box $[a_{i_1}, b_{i_1}] \times [a_{i_2}, b_{i_2}] \times \cdots \times [a_{i_d}, b_{i_d}] \times [0, 1]$ gives us a box representation of G in $d+1$ dimensions. Extending this construction gives us a box representation of G in $d+e$ dimensions. Similarly, we can extend a unit cube representation to higher dimensions. □

PROPOSITION 2.7. *If G has t components $C_1, C_2, \ldots, C_t$, then*

$$box(G) = \max_{1 \leq i \leq t} box(C_i) \qquad \textit{and} \qquad cub(G) = \max_{1 \leq i \leq t} cub(C_i)$$

Proof. By Proposition 2.6, we can construct a box representation of each component in $\max_{1 \leq i \leq t} box(C_i) = x$ dimensions. Now, since each component has a finite number of vertices, we can translate the boxes of each component sufficiently far apart along one dimension so that they don't intersect. This is a box representation of G in x dimensions. □

2.4 Dimensional and Codimensional Properties of Graphs

2.4.1 Dimensional Properties

DEFINITION 2.6 ([11]). *A property P of graphs is a dimensional property if every graph can be represented as the intersection of graphs, each satisfying property P.*

If P is a dimensional property, let $d_P(G)$ be the least integer k such that G is the intersection of k graphs, each satisfying property P.

Let $I(G)$ denote the property of G being an interval graph and $U(G)$ denote the property of G being a unit interval graph.

THEOREM 2.8 ([22]). *$d_I(G) = box(G)$.*

Proof. First, we prove that $d_I(G) \leq box(G)$. Consider a box representation of G in $box(G)$ dimensions. Projecting each of these boxes on the coordinate axes gives us $box(G)$ interval supergraphs - say $I_1, I_2, \ldots, I_{box(G)}$ - of G. Moreover, for $(a, b) \notin E(G)$, $\exists\, i, 1 \leq i \leq box(G)$ such that $(a, b) \notin E(I_i)$, else the boxes corresponding to a and b would intersect. So, we have $box(G)$ interval graphs - $I_1, I_2, \ldots, I_{box(G)}$ such that $I_1 \cap I_2 \cap \cdots \cap I_{box(G)} = G$.

Now, we prove that $box(G) \leq d_I(G)$. $d_I(G)$ is the least integer k such that G is the intersection of k interval graphs, i.e. there exist interval graphs $I_1, I_2, \ldots, I_{d_I(G)}$ such that $I_1 \cap I_2 \cap \cdots \cap I_{d_I(G)} = G$. Now, for each vertex v, we construct a box as the cartesian product of the $d_I(G)$ intervals (interval associated with v in each of the $d_I(G)$ interval graphs). If $(u, v) \in E(G)$, the boxes corresponding to u and v intersect since the intervals corresponding to u and v intersect in each interval graph I_i, $1 \leq i \leq d_I(G)$. Conversely, if $(u, v) \notin E(G)$, $\exists\, i, 1 \leq i \leq d_I(G)$ such that the intervals associated with u and v in I_i do not intersect. Consequently, the boxes associated with u and v do not intersect. Thus, we have demonstrated a box representation of G in $d_I(G)$ dimensions. $\square$

Similarly, we have the following equivalence:

THEOREM 2.9. *$d_U(G) = cub(G)$.*

In Section 1.2, we had mentioned that $box(C^4) = 2$.

PROPOSITION 2.10. *$box(C^4) = 2$.*

Proof. First we note that $box(C^4) \leq 2$ since C^4 has a box representation in 2 dimensions as shown in figure 1.6, Section 1.2. Now, by Lemma 2.2, C^4 is not an interval graph. So, by Theorem 2.8 $box(C^4) > 1$. $\square$

We now use Theorem 2.8 to give an upper bound on the boxicity of a graph.

LEMMA 2.11. *The boxicity of the Hajós graph*[1] *denoted by S_2 (shown in figure 2.1) is 2.*

[1] *The Hajós graph is also known as the* 2*-sun graph or the Sierpiński graph of order* 2.

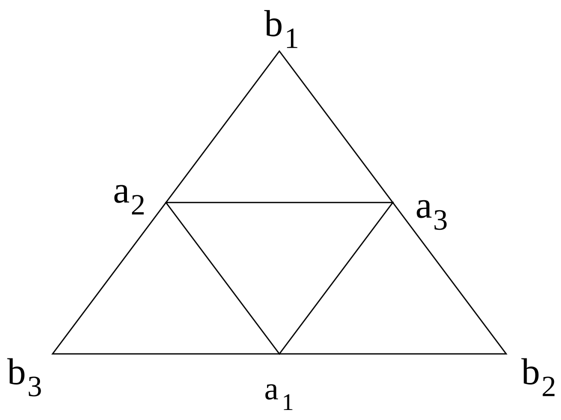

Figure 2.1: The Hajós graph

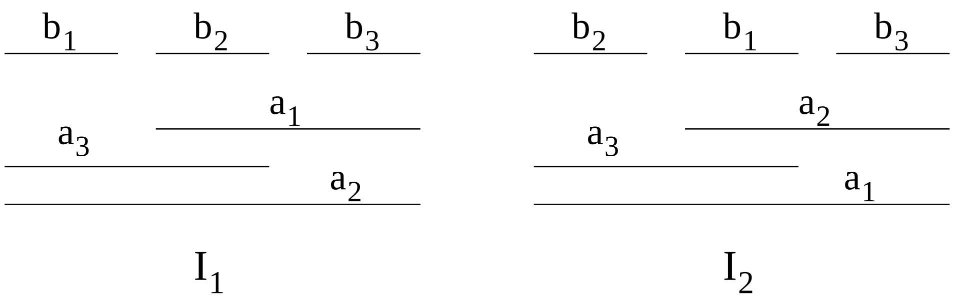

Figure 2.2: Interval supergraphs of the Hajós graph

Proof. First, we claim that the Hajós graph is not an interval graph. Assume for the sake of contradiction that the Hajós graph has an interval representation I_{S_2}. Since $\{b_1, b_2, b_3\}$ is an independent set, the intervals corresponding to these 3 vertices do not overlap each other. Without loss of generality, we may assume that $l(b_1) \leq r(b_1) < l(b_2) \leq r(b_2) < l(b_3) \leq r(b_3)$. Now, $(a_2, b_1) \in E(S_2) \Rightarrow l(a_2) \leq r(b_1)$ and $(a_2, b_3) \in E(S_2) \Rightarrow l(b_3) \leq r(a_2)$. Then, $l(b_2) \in [l(a_2), r(a_2)]$ which is a contradiction since $(a_2, b_2) \notin E(S_2)$. Thus, $box(S_2) \geq 2$.

Now we prove that $box(S_2) \leq 2$. Let I_1 and I_2 be the interval graphs shown in figure 2.2. Since $S_2 = I_1 \cap I_2$, by Theorem 2.8 we can say that $box(S_2) \leq 2$. Alternatively, to prove that $box(S_2) \leq 2$, we just need to show a box representation of S_2 in 2 dimensions as in figure 2.3. ∎

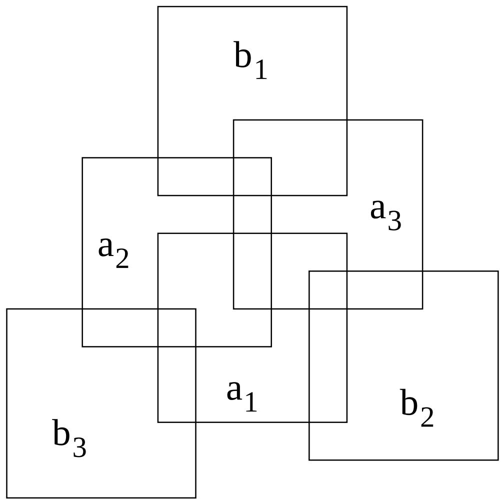

Figure 2.3: A box representation of the Hajós graph

2.4.2 Codimensional Properties

Definition 2.7. *A property P of graphs is a codimensional property if every graph can be represented as the union of graphs, each satisfying property P.*

If P is a codimensional property, let $c_P(G)$ be the least integer k such that G is the union of k graphs, each satisfying property P.

Let $\overline{P}$ be the property which holds for G if and only if P holds for $\overline{G}$.

Proposition 2.12 ([11]). *P is dimensional if and only if $\overline{P}$ is codimensional. If P is dimensional then, $d_P(G) = c_{\overline{P}}(\overline{G})$.*

Proof. First we show that if P is dimensional, then $\overline{P}$ is codimensional and $c_{\overline{P}}(\overline{G}) \leq d_P(G)$. Since P is dimensional, every graph G can be expressed as the intersection of $d_P(G)$ graphs, each satisfying property P. Say $G = P_1 \cap P_2 \cap \cdots \cap P_{d_P(G)}$. By De Morgan's law, $\overline{G} = \overline{P}_1 \cup \overline{P}_2 \cup \cdots \cup \overline{P}_{d_P(G)}$. Thus, $\overline{G}$ can be expressed as the union of $d_P(G)$ graphs, each satisfying property $\overline{P}$. i.e., $c_{\overline{P}}(\overline{G}) \leq d_P(G)$.

Similarly, it can be shown that if $\overline{P}$ is codimensional, then P is dimensional and $d_P(G) \leq c_{\overline{P}}(\overline{G})$. □

The complement of an interval graph is called a co-interval graph and the complement of a unit interval graph is called a co-unit interval graph. As examples of codimensional properties, $\overline{I}(G)$ denotes the property of G being a co-interval graph and $\overline{U}(G)$ denotes

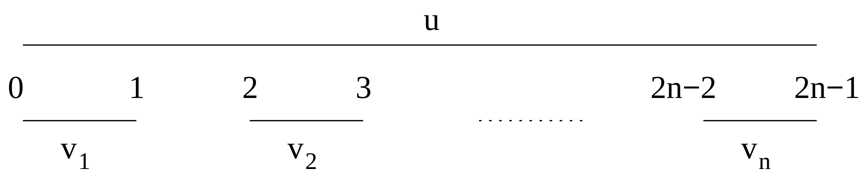

Figure 2.4: An interval representation of $K_{1,n}$

the property of G being a co-unit interval graph.

2.5 Relationship between Boxicity and Cubicity

In this section, we shall find the boxicity and cubicity of $K_{1,r}$ and give a brief review of the existing literature on boxicity.

LEMMA 2.13. $box(K_{1,r}) = 1$; $r \geq 2$.

Proof. Let $V(K_{1,r}) = \{u, v_1, v_2, \ldots, v_r\}$ and $d(u) = r$. Let $l(v_i) = 2 \cdot (i-1)$, $r(v_i) = 2 \cdot i - 1$ for $1 \leq i \leq r$ and $l(u) = 0$, $r(u) = 2 \cdot r - 1$ as shown in figure 2.4. Then, $[l(v_i), r(v_i)] \cap [l(v_j), r(v_j)] = \emptyset, \forall\ i, j; 1 \leq i, j \leq r$ and $[l(v_i), r(v_i)] \cap [l(u), r(u)] = [l(v_i), r(v_i)], \forall\ i, 1 \leq i \leq r$. Since $(v_i, v_j) \notin E(K_{1,r}), \forall\ i, j; 1 \leq i, j \leq r$ and $(u, v_i) \in E(K_{1,r}), \forall\ i, 1 \leq i \leq r$, we have given an interval representation of $K_{1,r}$. i.e., $box(K_{1,r}) \leq 1$. Since $r \geq 2$, $K_{1,r} \neq K_s$ for any s and $box(K_{1,n}) \geq 1$. $\square$

LEMMA 2.14. $cub(K_{1,3}) = 2$.

Proof. Let $V(K_{1,3}) = \{u, v_1, v_2, v_3\}$ and $d(u) = 3$. To prove that $cub(K_{1,3}) \leq 2$, we need to show a unit cube representation of $K_{1,3}$ in $\mathbb{R}^2$. $f(u) = (0,0), f(v_1) = (1,1), f(v_2) = (1,-1)$ and $f(v_3) = (-1,1)$ is the desired unit cube representation.

Now, assume for the sake of contradiction that $cub(K_{1,3}) = 1$. Without loss of generality, let $f(v_1) \leq f(v_2) \leq f(v_3)$. Then, since $(v_1, v_2) \notin E(K_{1,3}), f(v_2) - f(v_1) > 1$ and since $(v_2, v_3) \notin E(K_{1,3}), f(v_3) - f(v_2) > 1$. So, $f(v_3) - f(v_1) > 2$. But, $(u, v_1) \in E(K_{1,3}) \Rightarrow |f(u) - f(v_1)| \leq 1$ and $(u, v_3) \in E(K_{1,3}) \Rightarrow |f(u) - f(v_3)| \leq 1$. Then, $f(v_3) - f(v_1) \leq |f(v_3) - f(u)| + |f(u) - f(v_1)| \leq 2$ which is a contradiction. $\square$

THEOREM 2.15 ([22]). $cub(K_{1,r}) = \lceil \log r \rceil$; $r \geq 1$.

Proof. Let $V(K_{1,r}) = \{u, v_1, v_2, \ldots, v_r\}$ and $d(u) = r$. First, we give a unit cube representation of $K_{1,r}$ in $\lceil \log r \rceil$ dimensions to show that $cub(K_{1,r}) \leq \lceil \log r \rceil$. Let the cube corresponding to u be at $(0, 0, \ldots, 0)$ and the cubes corresponding to vertices $v_1, v_2, \ldots, v_r$ be at distinct $(-1, 1)$ vectors. Note that for $\lceil \log r \rceil$ dimensions, we can get r such distinct vectors. Now, $\rho(v_i, v_j) = 2$, where $1 \leq i, j \leq r$, $i \neq j$ and $\rho(u, v_i) = 1$, $1 \leq i \leq r$.

Now we prove that $cub(K_{1,r}) \geq \lceil \log r \rceil$. In other words, we prove that $K_{1,r}$ does not have a unit cube representation in $\mathbb{R}^p$ if $r > 2^p$ by induction on p. For $p = 0$, $r > 1$ implies that $K_{1,2}$ is an induced subgraph of $K_{1,r}$ and since $cub(K_{1,2}) = 1$, by Proposition 2.5 we have $cub(K_{1,r}) \geq 1$. For $p = 1$, $r \geq 2$ implies that $K_{1,3}$ is an induced subgraph of $K_{1,r}$. By Lemma 2.14, we know that $cub(K_{1,3}) = 2$. So, by Proposition 2.5, $cub(K_{1,r}) \geq 2$. This is the basis for our induction. We now assume that for $p \geq 1$, $K_{1,r}$ does not have a unit cube representation in $\mathbb{R}^p$ if $r > 2^p$ and show that $K_{1,r}$ does not have a unit cube representation in $\mathbb{R}^{p+1}$ if $r > 2^{p+1}$. For the sake of contradiction assume that for $r > 2^{p+1}$ there exists a unit cube representation of $K_{1,r}$ in $\mathbb{R}^{p+1}$. Let $f : V \rightarrow \mathbb{R}^{p+1}$ be a function which maps each vertex of $K_{1,r}$ to an $n + 1$ dimensional unit cube. Let $f_i(v)$ denote the i-th element of the $(p+1)$-tuple $f(v)$. Now, let $S_1 = \{v_i : f_{p+1}(v_i) \geq f_{p+1}(u)\}$ and $S_2 = \{v_i : f_{p+1}(v_i) \leq f_{p+1}(u)\}$. Without loss of generality, we may assume that $|S_1| \geq |S_2|$. Thus, $|S_1| > 2^p$. Let g_i be the restriction of f_i to $S_1 \cup \{u\}$ for $1 \leq i \leq p$. Then, $g_1, g_2, \ldots, g_p$ is a representation of $K_{1,|S_1|}$ in $\mathbb{R}^p$, which contradicts our assumption.
□

In [4] it is shown that $cub(G) \leq \lceil \log n \rceil \cdot box(G)$. This equality holds tight for $K_{1,n-1}$ as shown by Lemma 2.13 and Theorem 2.15 . We know from Proposition 2.4 that $box(G) \leq cub(G)$. Thus,

$$box(G) \leq cub(G) \leq \lceil \log n \rceil \cdot box(G). \tag{2.2}$$

From Lemmas 2.13 and 2.14, we know that $K_{1,3}$ is an interval graph but not a unit interval graph. In [19] it was shown that a graph is a unit interval graph if and only if it is an interval graph and has no induced $K_{1,3}$.

It was shown in [9] that computing the boxicity of a graph is NP-hard. In fact, deciding whether the boxicity of a graph is at most 2 is also NP-complete [16]. The boxicity of a graph is at most $2D^2$ [3], where D is the maximum degree of the graph. Thomassen [26] proved that the boxicity of planar graphs is at most 3. Scheinerman [23] showed that the boxicity of outer planar graphs is at most 2. The cubicity of the d-dimensional hypercube is $\Theta\left(\frac{d}{\log d}\right)$ [5].

Researchers have also tried to extend the concept of boxicity in various ways. The circular dimension [14, 24], rectangle number [29] and grid dimension [1] are some examples.

Chapter 3

Boxicity

3.1 Introduction

In this chapter, we shall see bounds on the boxicity of some graphs. We can show an upper bound of d on the boxicity (cubicity) of a graph, by demonstrating a box (unit cube) representation of the graph in d dimensions or by representing the graph as the intersection of d interval (unit interval) graphs. In Section 3.2 we shall see examples of how to prove lower bounds and upper bounds on the boxicity of a graph. In Section 3.3 we see a technique of proving an upper bound on the boxicity of a graph by expressing the complement of the graph as the union of co-interval graphs [1]. In Section 3.5 we give our proof that if G is a bipartite graph, $box(G) \leq \lceil \frac{n}{4} \rceil$ and in Section 3.4, we give our proof that for any graph G, $box(G) \leq \lfloor \frac{t}{2} \rfloor + 1$ where t is the cardinality of a minimum vertex cover. In Section 3.6, we give our proof that if $box(G) = \frac{n}{2} - s$ where $s \geq 0$, then $\chi(G) \geq \frac{n}{2s+2}$, where $\chi(G)$ is the chromatic number of G. We interchangeably refer to an interval representation of a graph as the interval graph itself.

[1] A co-interval graph is the complement of an interval graph

3.2 Bounds on the Boxicity of some Graphs

In this Section, we see upper bounds on the boxicity of a tree and a split graph. We also see the boxicity of $K_2^{\frac{n}{2}}$ (when n is even).

3.2.1 An Upper Bound on the Boxicity of a Tree

LEMMA 3.1. *For a tree T, $box(T) \leq 2$.*

Proof. We shall show that T can be represented as the intersection of 2 interval graphs I_1 and I_2 and by Theorem 2.8 this would imply that $box(T) \leq 2$. Arbitrarily, choose a vertex r as the root of the tree.

In I_1, we associate with each vertex v the interval $[d_v, d_v + 1]$, where d_v is the depth of vertex v. Thus,

$$(u, v) \in E(I_1) \Leftrightarrow d_u = d_v \text{ or } d_u = d_v + 1 \text{ or } d_u = d_v - 1. \tag{3.1}$$

In I_2, we let the intervals correspond to timestamps in a DFS (Depth-First Search) of T as explained in [25]. The first timestamp of a vertex v corresponds to $l(v)$ and the second timestamp corresponds to $r(v)$. By the Parenthesis theorem [25], exactly one of the following 3 cases holds for 2 vertices u and v:

Case 1: The intervals $[l(u), r(u)]$ and $[l(v), r(v)]$ are entirely disjoint and neither u nor v is a descendant of the other in the DFS tree.

Case 2: The interval $[l(u), r(u)]$ is entirely contained within the interval $[l(v), r(v)]$ and u is a descendant of v in the DFS tree.

Case 3: The interval $[l(v), r(v)]$ is entirely contained within the interval $[l(u), r(u)]$ and v is a descendant of u in the DFS tree.

So, $(u, v) \in E(I_2)$ if and only if Case 2 or 3 above holds. But by equation 3.1, $(u, v) \in E(I_1 \cap I_2)$ if and only if u is the parent or a child of v. Thus, $I_1 \cap I_2 = T$. □

In [6] it is shown that $box(G) \leq tw(G) + 2$, where $tw(G)$ is the treewidth of G.

3.2.2 The Boxicity of $K_2^{\frac{n}{2}}$

We shall see in Section 3.3 (Theorem 3.11) and Section 3.4 (Corollary 3.17) that $box(G) \leq \lfloor \frac{n}{2} \rfloor$. This inequality is tight for $T(n, \frac{n}{2}) = \overline{\frac{n}{2}K_2} = K_2^{\frac{n}{2}}$, the Turan graph with $\frac{n}{2}$ partitions, each of size 2.

LEMMA 3.2. $box(T(n, \frac{n}{2})) = \frac{n}{2}$.

Proof. Let $A = \{a_1, a_2, \ldots, a_{\frac{n}{2}}\}$ and $B = \{b_1, b_2, \ldots, b_{\frac{n}{2}}\}$ be the vertices of the graph such that $(a_i, b_i) \notin E(T(n, \frac{n}{2}))$. All the other pairs of vertices are adjacent to each other.

First we show that $box(T(n, \frac{n}{2})) \leq \frac{n}{2}$ by constructing $\frac{n}{2}$ interval graphs $I_1, I_2, \ldots, I_{\frac{n}{2}}$.

Construction of I_i for $1 \leq i \leq \frac{n}{2}$:

In (the interval representation of) I_i, we map each vertex $v \in V(T(n, \frac{n}{2}))$ to an interval $f_i(v)$ as follows:

$$
\begin{aligned}
f_i(v) &= [0, 1] && if\ v = a_i \\
&= [2, 3] && if\ v = b_i \\
&= [1, 2] && if\ v \in V(T(n, n/2)) - \{a_i, b_i\}.
\end{aligned}
$$

Then, $(a_i, b_i) \notin E(I_i)$. For $u, v \neq a_i, b_i$, $(a_i, v) \in E(I_i)$ since $1 \in f_i(a_i)$ and $1 \in f_i(v)$, $(b_i, v) \in E(I_i)$ since $2 \in f_i(b_i)$ and $2 \in f_i(v)$, $(u, v) \in E(I_i)$ since $1 \in f_i(u)$ and $1 \in f_i(v)$. So, $E(I_i) = E(K^n) - \{(a_i, b_i)\}$. It follows that $I_1 \cap I_2 \cap \cdots \cap I_n = T(n, \frac{n}{2})$. By Theorem 2.8, $box(T(n, \frac{n}{2})) \leq \frac{n}{2}$.

Now we show that $box(T(n, \frac{n}{2})) \geq \frac{n}{2}$. For the sake of contradiction, assume that $box(T(n, \frac{n}{2})) = l < \frac{n}{2}$. By Theorem 2.8, there are l interval graphs say $I_1, I_2, \ldots, I_l$ such that $I_1 \cap I_2 \cap \cdots \cap I_l = T(n, \frac{n}{2})$. Since $\frac{n}{2}$ pairs of vertices are nonadjacent in $T(n, \frac{n}{2})$, at least one of these l interval graphs has two distinct pairs of nonadjacent vertices. Let I_k, $1 \leq k \leq l$ be an interval graph having two pairs of nonadjacent vertices, say (c_1, d_1) and (c_2, d_2). Now, since $E(I_k) \supseteq E(T(n, \frac{n}{2}))$, the four vertices c_1, c_2, d_1, d_2 induce a C^4 in I_k and by Proposition 2.10, the boxicity of C^4 is 2. By Proposition 2.5, $box(I_k) \geq 2$ which is a contradiction since I_k is an interval graph. □

3.2.3 An Upper Bound on the Boxicity of a Split Graph

Let G be a split graph with vertex partition $V = K \uplus S$ where K is a clique and S is an independent set. Let $K = \{v_1, v_2, \ldots, v_p\} \neq \emptyset$ and $S = \{u_1, u_2, \ldots, u_q\}$. We construct $\lceil \frac{p}{2} \rceil$ interval graphs $I_1, I_2, \ldots, I_{\lceil \frac{p}{2} \rceil}$ and show that $G = I_1 \cap I_2 \cap \cdots \cap I_{\lceil \frac{p}{2} \rceil}$.

Construction of I_i for $1 \leq i \leq \lfloor \frac{p}{2} \rfloor$:

We map each $v \in V$ to an interval $f_i(v)$ on the real line as follows:

$$\begin{aligned}
f_i(v) &= [0,2] \quad if \;\; v = v_{2i-1}. \\
&= [1,3] \quad if \;\; v = v_{2i}. \\
&= [0,4] \quad if \;\; v \in K - \{v_{2i-1}, v_{2i}\} \\
&= \left[\frac{2j-1}{2q+1}, \frac{2j}{2q+1}\right] \quad if \;\; v = u_j \in N_G(v_{2i-1}) - N_G(v_{2i}). \\
&= \left[1 + \frac{2j-1}{2q+1}, 1 + \frac{2j}{2q+1}\right] \quad if \;\; v = u_j \in N_G(v_{2i}) \cap N_G(v_{2i-1}). \\
&= \left[2 + \frac{2j-1}{2q+1}, 2 + \frac{2j}{2q+1}\right] \quad if \;\; v = u_j \in N_G(v_{2i}) - N_G(v_{2i-1}). \\
&= \left[3 + \frac{2j-1}{2q+1}, 3 + \frac{2j}{2q+1}\right] \quad if \;\; v = u_j \notin \{N_G(v_{2i}) \cup N_G(v_{2i-1}\}).
\end{aligned}$$

Claim 1. *For each i, $1 \leq i \leq \lfloor \frac{p}{2} \rfloor$, $E(G) \subseteq E(I_i)$.*

Proof. Let (a,b) be an edge of G. If $\{a,b\} \subseteq K$, then $1 \in f_i(a)$ and $1 \in f_i(b)$. So $(a,b) \in E(I_i)$. Otherwise either $a \in S$ or $b \in S$, but $\{a,b\} \subseteq S$ is not possible. Without loss of generality, we may assume that $a \in K$ and $b \in S$. Note that $0 < \frac{2j-1}{2q+1} < \frac{2j}{2q+1} < 1$ when $1 \leq j \leq q$. So, due to the mapping of the interval to b, $(a,b) \in E(I_i)$. □

Construction of $I_{\lceil \frac{p}{2} \rceil}$ when p is odd:

For this interval, we map each $v \in V$ to an interval $f_{\lceil \frac{p}{2} \rceil}(v)$ on the real line as follows:

$$\begin{aligned}
f_{\lceil \frac{p}{2} \rceil}(v) &= [0,1] \quad if \;\; v = v_{\lceil \frac{p}{2} \rceil}. \\
&= [0,2] \quad if \;\; v \in K - \{v_{\lceil \frac{p}{2} \rceil}\} \\
&= \left[\frac{2j-1}{2q+1}, \frac{2j}{2q+1}\right] \quad if \;\; v = u_j \in N_G(v_{\lceil \frac{p}{2} \rceil}).
\end{aligned}$$

$$= \left[1 + \frac{2j-1}{2q+1}, 1 + \frac{2j}{2q+1}\right] \qquad if \quad v = u_j \notin N_G(v_{\lceil \frac{p}{2} \rceil}).$$

Claim 2. $E(G) \subseteq E(I_{\lceil \frac{p}{2} \rceil})$.

Proof. Let (a, b) be an edge of G. If $\{a, b\} \subseteq K$, then $0 \in f_{\lceil \frac{p}{2} \rceil}(a)$ and $0 \in f_{\lceil \frac{p}{2} \rceil}(b)$. So $(a, b) \in E(I_i)$. Otherwise either $a \in S$ or $b \in S$, but $\{a, b\} \subseteq S$ is not possible. Without loss of generality, we may assume that $a \in K$ and $b \in S$. Note that $0 < \frac{2j-1}{2q+1} < \frac{2j}{2q+1} < 1$ when $1 \leq j \leq q$. So, due to the mapping of the interval to b, $(a, b) \in E(I_i)$. □

The following Lemma follows from Claims 1 and 2.

Lemma 3.3. *For each interval graph I_i, $1 \leq i \leq \lceil \frac{p}{2} \rceil$, $E(G) \subseteq E(I_i)$.*

Lemma 3.4. *For any $(x, y) \notin E(G)$, there exists some $i, 1 \leq i \leq \lceil \frac{p}{2} \rceil$, such that $(x, y) \notin E(I_i)$.*

Proof. Suppose $(x, y) \notin E(G)$. First note that $\{x, y\} \subseteq K$ is not possible.

Case 1: $\{x, y\} \subseteq S$.

Let $x = u_k$ and $y = u_l$, where $k \neq l$. It is easy to see that $f_i(x) \cap f_i(y) = \emptyset \; \forall\, i, 1 \leq i \leq \lceil \frac{p}{2} \rceil$. Since $K \neq \emptyset, p \geq 1$ and $\lceil \frac{p}{2} \rceil \geq 1$. Hence $(x, y) \notin E(I_1)$.

Case 2: $\{x, y\} \cap K \neq \emptyset$

Without loss of generality, we may assume that $x = v_k \in K$ and $y = u_l \in S$. Then, by our construction, $f_{\lceil \frac{k}{2} \rceil}(x) \cap f_{\lceil \frac{k}{2} \rceil}(y) = \emptyset$. Thus, $(x, y) \notin E(I_{\lceil \frac{k}{2} \rceil})$. □

By Lemmas 3.3 and 3.4, we get $E(G) = E(I_1) \cap E(I_2) \cap \cdots \cap E(I_{\lceil \frac{p}{2} \rceil})$. Thus by Theorem 2.8, we obtain the following:

Theorem 3.5 ([10]). *For a split graph G with vertex partition $V = K \uplus S$, $box(G) \leq \left\lceil \frac{|K|}{2} \right\rceil$ if $K \neq \emptyset$.*

3.3 Covering a Graph by Co-interval Graphs

In Chapter 2.4, we introduced a general theory of dimensional and codimensional properties. Here, we shall use Proposition 2.12 to show upper bounds on the boxicity of some graphs.

Lemma 3.6 ([11]). *If G is a co-interval graph, then the union of G and an isolated vertex is a co-interval graph.*

Proof. Let $J = \overline{G}$ be an interval graph and let H be the graph obtained from J by adding a vertex u which is adjacent to all the vertices of J. Now consider an interval representation of J and add another interval corresponding to vertex u of H which spans all the intervals. This is an interval representation of H. So, by Proposition 2.12, $d_I(H) = 1 = c_{\overline{I}}(\overline{H})$. □

Corollary 3.7. *If G is a co-interval graph, then the union of G and isolated vertices is a co-interval graph.*

Definition 3.1. *A set $\mathcal{C}$ of spanning subgraphs of a graph G is an edge covering of G if each edge of G belongs to some $C \in \mathcal{C}$.*

Lemma 3.8 ([10]). *$box(G) \leq k$ if and only if there is an edge covering $\mathcal{C}$ of $\overline{G}$ such that $|\mathcal{C}| = k$ and each graph in $\mathcal{C}$ is a co-interval spanning subgraph of $\overline{G}$.*

Proof. For $k = 0$, $G = K^n$ and $\mathcal{C} = \emptyset$. For $k \geq 1$, by Theorem 2.8 we know that $box(G) \leq k$ if and only if $G = I_1 \cap I_2 \cap \cdots \cap I_k$ where each I_i is an interval graph. By De Morgan's law, $\overline{G} = \overline{I}_1 \cup \overline{I}_2 \cup \cdots \cup \overline{I}_k$. But, this is equivalent to saying that $\overline{G}$ has an edge covering $\mathcal{C} = \{\overline{I}_1, \overline{I}_2, \ldots, \overline{I}_k\}$ where $\overline{I}_i$, $1 \leq i \leq k$ is a co-interval graph. □

Definition 3.2. *For 2 adjacent vertices u and v, the u, v ant of $G(V, E)$ represented by $(u - v)(G)$ is the spanning subgraph of G whose vertex set is $V(G)$ and whose edge set is $\{(u, v)\} \cup \{(u, w) | (u, w) \in E(G)\} \cup \{(v, w) | (v, w) \in E(G)\}$.*

Lemma 3.9. *([10]) If $(u, v) \in E(G)$, then $(u - v)(G)$ is a co-interval graph.*

Proof. We show that $\overline{(u - v)(G)} = H$ is an interval graph and it will follow that $(u - v)(G)$ is a co-interval graph. To show that H is an interval graph, we map each vertex $x \in H$ to an interval $f(x)$ as follows:

$$
\begin{aligned}
f(x) &= [0, 1] && if\ x = u \\
&= [4, 5] && if\ x = v \\
&= [2, 3] && if\ (x, u), (x, v) \notin E(H)\ and\ x \neq u, v \\
&= [0, 3] && if\ (x, u) \in E(H), (x, v) \notin E(H)\ and\ x \neq u, v
\end{aligned}
$$

$$
\begin{aligned}
&= [2,5] \qquad if\ (x,u) \notin E(H), (x,v) \in E(H)\ and\ x \neq u, v \\
&= [0,5] \qquad if\ (x,u), (x,v) \in E(H)\ and\ x \neq u, v.
\end{aligned}
$$

For $\{a, b\} \subseteq V(H)$, one of the following cases holds true:

Case 1: $a, b \neq u, v$.

$f(a) \cap f(b) \subseteq [2,3] \neq \emptyset$ and $(a,b) \in E(H)$ since $(a,b) \notin E((u-v)(G))$.

Case 2: $a = u; b \neq u, v$.

$f(a) \cap f(b) \neq \emptyset \Leftrightarrow (a,b) \in E(H)$.

Case 3: $a = v; b \neq u, v$.

$f(a) \cap f(b) \neq \emptyset \Leftrightarrow (a,b) \in E(H)$.

Case 4: $a = v; b = v$.

$f(a) \cap f(b) = \emptyset$ and $(a,b) \notin E(H)$ since $(a,b) \in E((u-v)(G))$.

Thus, we have demonstrated an interval representation of H. □

LEMMA 3.10. *$box(\overline{C^n}) = \lceil \frac{n}{3} \rceil$, $n \geq 5$.*

Proof. Let $v_1 v_2 \ldots v_n v_1$ be the cycle C^n. First we show that $box(\overline{C^n}) \leq \lceil \frac{n}{3} \rceil$. Let $\lceil \frac{n}{3} \rceil = r$ and let Q_i be the path $v_{3i-2} v_{3i-1} v_{3i} v_{3i+1}$ for $1 \leq i < r$ and Q_r be the path $v_{n-2} v_{n-1} v_n v_1$. Note that a path $v_1 v_2 v_3 v_4$ is a v_2, v_3 ant and hence a co-interval graph. By Corollary 3.7, the union of such a path and isolated vertices is also a co-interval graph. Thus, all the edges of C^n are covered by the $\lceil \frac{n}{3} \rceil$ co-interval graphs with edge sets same as $Q_1, Q_2, \ldots, Q_r$ respectively, each of which is the union of a path of length 3 and isolated vertices.

Now, to show that $box(\overline{C^n}) \geq \lceil \frac{n}{3} \rceil$, assume that $box(\overline{C^n}) \leq \lceil \frac{n}{3} \rceil - 1$ for the sake of contradiction. Then, by Theorem 2.8, there exist $box(\overline{C^n})$ interval graphs $I_1, I_2, \ldots, I_{box(\overline{C^n})}$ such that $\overline{C^n} = I_1 \cap I_2 \cap \cdots \cap I_{box(\overline{C^n})}$. By the pigeonhole principle, there exists at least 1 interval graph I_k such that $||\overline{I_k} \cap C^n|| \geq 4$. But then, $\overline{\overline{I_k} \cap C^n}$ induces a C^4 in $\overline{C^n}$ which is a contradiction, since $box(C^4) = 2$ and by Proposition 2.5 this means that $box(I_k) \geq 2$. □

THEOREM 3.11 ([10]). *$box(G) \leq \lfloor \frac{n}{2} \rfloor$, for any graph G.*

Proof. If $|E(\overline{G})| = 0$, $G = K^n$ and $box(G) = 0$. Otherwise choose $(a_1, b_1) \in E(\overline{G})$. Let $C_1 = (a_1 - b_1)(\overline{G})$. If $\mathcal{C} = \{C_1\} = \{\overline{G}\}$, then by Lemma 3.9, $box(G) = 1 \leq \lfloor \frac{n}{2} \rfloor$ since $n \geq 2$. Otherwise keep choosing an edge of $\overline{G}$, say (a_{i+1}, b_{i+1}) not covered by $\mathcal{C} = \bigcup_{1 \leq j \leq i} \{C_j\}$. Let $C_{i+1} = (a_{i+1} - b_{i+1})(\overline{G})$. Suppose all edges of $\overline{G}$ are covered by $\mathcal{C} = \bigcup_{1 \leq j \leq l} \{C_j\}$ when we terminate this procedure. Note that $a_1, b_1, a_2, b_2, \ldots, a_l, b_l$ are all distinct and we have covered $\overline{G}$ by l ants. Thus, by Lemmas 3.9 and 3.8, $box(G) \leq l = \frac{2l}{2} \leq \lfloor \frac{n}{2} \rfloor$. □

Trotter [28] has shown that when n is even, $T(n, \frac{n}{2})$ is the only graph which satisfies equality in the above equation. When n is odd, he again gives a characterization of graphs satisfying equality in the above equation. For $n = 5$, $\overline{C^5} = C^5$ and a graph containing $T(4, 2)$ as an induced subgraph are the only graphs satisfying equality. For $n = 7$, the only graphs satisfying equality in the above equation are $\overline{C^7}$, $\overline{C^5} \Box T(2, 1)$ or a graph containing $T(6, 3)$ as an induced subgraph.

Let G be a split graph with vertex partition $V = K \uplus S$, where K is a clique and S is an independent set.

Lemma 3.12 ([10]). *For a split graph G with vertex partition $V = K \uplus S$, $box(G) \leq \left\lceil \frac{|S|}{2} \right\rceil$.*

Proof. $\overline{G}$ is also a split graph with $\overline{G}[S]$ a clique and $\overline{G}[K]$ an independent set. Let $V(\overline{G}[S]) = \{u_1, u_2, \ldots, u_q\}$. When $|S| = 0$, $G = K$ and $box(G) = 0$. When $|S| = 1$, let $S = \{a\}$. We show that $box(G) \leq 1$ or that G is an interval graph. For this, we map each vertex $x \in V$ to an interval $f(x)$ as follows:

$$
\begin{aligned}
f(x) &= [0, 1] && if\ x = a \\
&= [0, 3] && if\ x \neq a\ and\ (x, a) \in E(G) \\
&= [2, 3] && if\ x \neq a\ and\ (x, a) \notin E(G).
\end{aligned}
$$

It is easy to see that this is an interval representation of G. When $|S| \geq 2$, the edges of $\overline{G}$ are covered by $(u_1 - u_2)(\overline{G}), (u_3 - u_4)(\overline{G}), \ldots, (u_{q-1} - u_q)(\overline{G})$ if q is even and by $(u_1 - u_2)(\overline{G}), (u_3 - u_4)(\overline{G}), \ldots, (u_{q-2} - u_{q-1})(\overline{G}), (u_{q-1} - u_q)(\overline{G})$ if q is odd. In either case

(q is even or q is odd), by Lemmas 3.9 and 3.8, $box(G) \leq \left\lceil \frac{|S|}{2} \right\rceil$. □

By Theorem 3.5 and Lemma 3.12, we have the following:

THEOREM 3.13 ([10]). *For a split graph G with vertex partition $V = K \uplus S$, $box(G) \leq \min\left\{\left\lceil \frac{|K|}{2} \right\rceil, \left\lceil \frac{|S|}{2} \right\rceil\right\}$.*

3.4 Boxicity and Vertex Cover

Let $G = (V, E)$ be a graph and MVC be a minimum vertex cover of G. Let $A = V - MVC$. Clearly A is an independent set in G. Suppose $|MVC| = t$ and $\left\lfloor \frac{t}{2} \right\rfloor = t_1$. In this section we show that $box(G) \leq t_1 + 1$.

Let l be the largest integer such that there exist sets $P, Q \subseteq MVC$ such that $P = \{a_1, a_2, \ldots, a_l\}$, $Q = \{b_1, b_2, \ldots, b_l\}$, $P \cap Q = \emptyset$, and $(a_i, b_i) \notin E(G)$. Next, we construct $t_1 + 1$ different interval super graphs of G, say $I_1, I_2, \ldots, I_{t_1+1}$, as follows.

Construction of I_i for $1 \leq i \leq l$:

Recall that, for $1 \leq i \leq l$, $(a_i, b_i) \notin E$. For each pair (a_i, b_i), $1 \leq i \leq l$, we construct an interval graph I_i. To construct I_i, we map each $v \in V$ to an interval $f_i(v)$ on the real line as follows:

$$\begin{aligned}
f_i(v) &= [0,1] \quad if \;\; v = a_i. \\
&= [4,5] \quad if \;\; v = b_i. \\
&= [0,3] \quad if \;\; v \in N_G(a_i) - N_G(b_i). \\
&= [2,5] \quad if \;\; v \in N_G(b_i) - N_G(a_i). \\
&= [0,5] \quad if \;\; v \in N_G(a_i) \cap N_G(b_i). \\
&= [2,3] \quad if \;\; v \in V - (\{a_i, b_i\} \cup N_G(a_i) \cup N_G(b_i)).
\end{aligned}$$

CLAIM 3. *For each i, $1 \leq i \leq l$, $E(G) \subseteq E(I_i)$.*

Proof. It is easy to see that if $v \in MVC - \{a_i, b_i\}$, then $3 \in f_i(v)$. So, $MVC - \{a_i, b_i\}$ is a clique in each I_i. If $v \in N_G(a_i) \cup \{a_i\}$, then $0 \in f_i(v)$. So, $N_G(a_i) \subseteq N_{I_i}(a_i)$. Similarly, if $v \in N_G(b_i) \cup \{b_i\}$, then $5 \in f_i(v)$. That is, $N_G(b_i) \subseteq N_{I_i}(b_i)$. So, $E(G) \subseteq E(I_i)$, for each i, $1 \leq i \leq l$. □

Construction of I_i for $l+1 \leq i \leq t_1$ (assuming $t_1 \geq l+1$):

Let $C = MVC - \{P \cup Q\}$. Clearly C induces a clique in G by the maximality of l. Let $|C| = k' = t - 2l$. Since $t_1 = \left\lfloor \frac{t}{2} \right\rfloor$, we have $k' = t - 2l \geq 2$ and hence $|C| \geq 2$. Let $C = \{c_1, c_2, \ldots, c_{k'}\}$. If k' is even, then let $k'' = k'$, otherwise let $k'' = k' - 1$. Let $C' = \{c_1, c_2, \ldots, c_{k''}\}$. Clearly $C' \subseteq C$.

Let G' be the graph induced by $C' \cup A$ in G. As C' induces a clique and A induces an independent set in G, G' is a split graph. So by Theorem 3.13, $box(G') \leq \min\left\{ \left\lceil \frac{k''}{2} \right\rceil, \left\lceil \frac{|A|}{2} \right\rceil \right\} \leq \frac{k''}{2}$ (as k'' is even and $k'' \geq 2$). That is, G' is the intersection of at most $\frac{k''}{2}$ interval graphs, say $I'_1, I'_2, \ldots, I'_{\frac{k''}{2}}$, by Theorem 2.8. Note that $l + \frac{k''}{2} = \left\lfloor \frac{t}{2} \right\rfloor = t_1$. Let g_i be a function that maps each $v \in V(I'_i)$ to a closed interval on the real line such that I'_i, for each i, $1 \leq i \leq \frac{k''}{2}$, is the intersection graph of the family of intervals $\{g_i(v) : v \in V(I'_i)\}$. Now, let L_j and R_j be numbers on the real line such that $L_j \leq x$, for all $x \in \bigcup_{v \in V(I'_j)}(g_j(v))$ and $R_j \geq y$, for all $y \in \bigcup_{v \in V(I'_j)}(g_j(v))$. To construct I_i, $l+1 \leq i \leq t_1$, map each $v \in V(G)$ to a closed interval $f_{l+j}(v)$, $1 \leq j \leq \frac{k''}{2}$ on the real line as follows.

$$\begin{aligned} f_{l+j}(v) &= g_j(v) \qquad if\ v \in V(I'_j) = V(G) - (P \cup Q) - (C - C') \\ &= [L_j, R_j] \quad otherwise. \end{aligned}$$

Claim 4. *For each I_i, $l+1 \leq i \leq t_1$, $E(G) \subseteq E(I_i)$.*

Proof. By the construction of I_i, $l+1 \leq i \leq t_1$, it is easy to see that if $v \in P \cup Q \cup (C - C')$, then $L_j \in f_{l+j}(v)$, $1 \leq j \leq \frac{k''}{2}$. So, $P \cup Q \cup (C - C')$ induces a clique in each I_i, $l+1 \leq i \leq t_1$. Also, if $u \in P \cup Q \cup (C - C')$, then $(u, v) \in E(I_i)$, for each $v \in V(I_i) - \{P \cup Q \cup (C - C')\}$, by the definition of L_i and R_i. As the collection of interval graphs $I'_1, I'_2, \ldots, I'_{\frac{k''}{2}}$ is an interval graph representation of G', by Theorem 2.8, $E(G') \subseteq E(I'_j)$, $1 \leq j \leq \frac{k''}{2}$. But in I_{l+j}, $f_{l+j}(v) = g_j(v)$, for all $v \in V(I'_j)$, $1 \leq j \leq \frac{k''}{2}$. So, $E(G') \subseteq E(I_{l+j})$, $1 \leq j \leq \frac{k''}{2}$. Hence for each I_i, $l+1 \leq i \leq t_1$, $E(G) \subseteq E(I_i)$. □

Construction of I_{t_1+1}:

We construct the last interval graph I_{t_1+1} as follows. If k' is odd then suppose $C - C' = \{v\}$. So, $v \notin V(G')$. Let $MVC' = MVC$ if k' is even and $MVC' = MVC - \{v\}$ if k' is odd. Let $A = \{x_1, x_2, \ldots, x_r\}$, where $|A| = r$. Note that $A \neq \emptyset$. If k' is odd, then without loss of generality suppose $\{x_1, x_2, \ldots, x_s\} = A \cap N_G(v)$. Now, map each vertex x of G to an interval $f_{t_1+1}(x)$ on the real line as follows.

$$\begin{aligned} f_{t_1+1}(x) &= [2i-1, 2i] && if\ x \in A\ and\ x = x_i. \\ &= [1, 2r] && if\ x \in MVC'. \\ if\ k'\ is\ odd\ then\ f_{t_1+1}(v) &= [1, 2s] \end{aligned}$$

Claim 5. *$E(G) \subseteq E(I_{t_1+1})$.*

Proof. It is easy to see that if $x \in MVC$, then $1 \in f_{t_1+1}(x)$. So, MVC induces a clique in I_{t_1+1}. Also, if $x \in MVC' \cup \{x_i\}$, for some i, $1 \leq i \leq r$, then $2i \in f_{t_1+1}(x)$. That is, each $x_i \in A$ is adjacent to all the vertices of MVC'. If $x = x_i$, $1 \leq i \leq s$, then $2i \in f_{t_1+1}(x_i) \cap f_{t_1+1}(v)$. Thus $(x_i, v) \in E(I_{t_1+1})$ for each i, $1 \leq i \leq s$. That is, $N_G(v) \subseteq N_{I_{t_1+1}}(v)$. So, $E(G) \subseteq E(I_{t_1+1})$. □

The following Lemma follows from Claims 3, 4 and 5.

Lemma 3.14. *For each interval graph I_i, $1 \leq i \leq t_1 + 1$, $E(G) \subseteq E(I_i)$.*

Lemma 3.15. *For any $(x, y) \notin E(G)$, there exists some i, $1 \leq i \leq t_1 + 1$, such that $(x, y) \notin E(I_i)$.*

Proof. Suppose $(x, y) \notin E(G)$. As C induces a clique in G, both x and y cannot be present in C.

Case 1: $\{x, y\} \subseteq A$.

Let $x = x_i$ and $y = x_j$, where $i \neq j$. It is easy to see that $f_{t_1+1}(x) \cap f_{t_1+1}(y) = \emptyset$. Hence x is non-adjacent to y in I_{t_1+1}.

Case 2: $\{x, y\} \cap \{P \cup Q\} \neq \emptyset$

Without loss of generality suppose $x \in P \cup Q$. So, in I_i, for some i, $1 \leq i \leq l$, say I_k, $f_k(x) = [0, 1]$ or $f_k(x) = [4, 5]$. If $f_k(x) = [0, 1]$, then $f_k(y)$ is either $[2, 3]$, $[2, 5]$ or $[4, 5]$ and if $f_k(x) = [4, 5]$, then $f_k(y)$ is either $[0, 1]$, $[0, 3]$ or $[2, 3]$. In both the cases

$f_k(x) \cap f_k(y) = \emptyset$. Hence x is non-adjacent to y in I_k.

Case 3: $\{x, y\} \cap \{P \cup Q\} = \emptyset$

Now, it is easy to see that one of x or y, say x, will belong to $MVC - \{P \cup Q\}$, and y will belong to A. If $x \in C'$, then it is easy to see that $x, y \in V(G')$. As $I'_1, I'_2, \ldots, I'_{\frac{k''}{2}}$ is an interval graph representation of G', by Theorem 2.8, there exists k, $1 \leq k \leq \frac{k''}{2}$ such that $(x, y) \notin I'_k$. But in I_{l+k}, $f_{l+k}(v) = g_k(v)$, for all $v \in I'_k$. So, x and y are non-adjacent in I_{l+k}.

Next suppose $x \in C - C'$. Now, in I_{t_1+1}, $f_{t_1+1}(x) = [1, 2s]$ and as $y \notin N_x(G)$, $y = c_j$, where $j > s$. It is easy to see that $f_{t_1+1}(x) \cap f_{t_1+1}(y) = \emptyset$. So, x and y are non-adjacent in I_{t_1+1}.

Hence there exists some i, $1 \leq i \leq t_1 + 1$, such that $(x, y) \notin E(I_i)$. □

By Lemmas 3.14 and 3.15, we get $E(G) = E(I_1) \cap E(I_2) \cap \cdots \cap E(I_{t_1+1})$. Thus by Theorem 2.8, we obtain the following:

THEOREM 3.16. *([2]) For a graph G with vertex cover MVC, $box(G) \leq \lfloor \frac{t}{2} \rfloor + 1$, where $t = |MVC|$.*

Note that Theorem 3.11 follows from this result.

COROLLARY 3.17. *([22]) $box(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. If $G = K^n$, then $box(G) = 0$. Otherwise, $\exists$ a, b such that $(a, b) \notin E(G)$. Then, $V(G) - \{a, b\}$ is a vertex cover of G of cardinality $n - 2$. So, by Theorem 3.16, $box(G) \leq \left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor$. □

3.4.1 Tightness result

In this section we illustrate some graphs for which the bound given in Theorem 3.16 for boxicity is tight. For C^4, $|MVC| = 2$ and we know from Lemma 2.10 that $box(C^4) = 2$. So, $box(C^4) = \frac{|MVC|}{2} + 1$.

CLAIM 6. *For the graph $T(n, \frac{n}{2})$, the cardinality of minimum vertex cover is $n - 2$.*

Proof. Let $G = T(n, \frac{n}{2})$. Let $a, b \in V(G)$ be such that $(a, b) \notin E(G)$. It is easy to verify that $V - \{a, b\}$ is a vertex cover of G. Thus, $|MVC| \leq n - 2$. Now, if possible suppose $|MVC| \leq n - 3$. Let a, b, and c be the vertices which are not present in MVC.

By the construction of $T(n, \frac{n}{2})$ there will exist an edge in the induced subgraph of G on $\{a, b, c\}$. Clearly this edge is not adjacent to any of the vertex of MVC. This is a contradiction. Hence the claim is true. $\Box$

For $T(n, \frac{n}{2})$, $\left\lfloor \frac{|MVC|}{2} \right\rfloor + 1 = \left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \frac{n}{2}$ (as n is even), which equals the boxicity of $T(n, \frac{n}{2})$. Thus the bound of Theorem 3.16 is tight for $T(n, \frac{n}{2})$.

3.5 Boxicity of Bipartite Graphs

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph such that $|V_1| = n_1$ and $|V_2| = n_2$. Suppose $n_1 \leq n_2$ and $n_1 \geq 3$. In this section we show that for a bipartite graph G, $box(G) \leq \min\{\lceil \frac{n_1}{2} \rceil, \lceil \frac{n_2}{2} \rceil\}$.

It is easy to see that $|MVC| \leq n_1$ in G. So, by Theorem 3.16, $box(G) \leq \lfloor \frac{n_1}{2} \rfloor + 1$. If n_1 is odd, then $\left\lfloor \frac{n_1}{2} \right\rfloor + 1 = \left\lceil \frac{n_1}{2} \right\rceil$. So, $box(G) \leq \min\,\{\left\lceil \frac{n_1}{2} \right\rceil, \left\lceil \frac{n_2}{2} \right\rceil\}$.

Now assume that n_1 is even. By Theorem 3.16, $box(G) \leq \left\lfloor \frac{n_1}{2} \right\rfloor + 1 = \frac{n_1}{2} + 1$ (as n_1 is even). But, we need to show that $box(G) \leq \frac{n_1}{2}$. So that, $box(G) \leq \min\{\left\lceil \frac{n_1}{2} \right\rceil, \left\lceil \frac{n_2}{2} \right\rceil\}$.

Suppose n_1 is even. We construct $\frac{n_1}{2}$ interval super graphs of G, say $I_1, I_2, \ldots, I_{\frac{n_1}{2}}$, as follows.

Construction of I_i for $1 \leq i \leq \frac{n_1}{2} - 1$:

Let $x, y \in V_1$ and $V_1' = V_1 - \{x, y\}$. Note that $V_1' \neq \emptyset$ as $|V_1| \geq 3$. Let G_1' be the graph induced by $V_1' \cup V_2$ in G. Let G_1 be a graph such that $V(G_1) = V(G_1')$ and $E(G_1) = E(G_1') \cup \{(a, b) \mid a, b \in V_1'\}$. Clearly V_1' induces a clique and V_2 induces an independent set in G_1. So, G_1 is a split graph. Now, by Theorem 3.13, $box(G_1) \leq \min\left\{\left\lceil \frac{n_1-2}{2} \right\rceil, \left\lceil \frac{n_2}{2} \right\rceil\right\} = \left\lceil \frac{n_1-2}{2} \right\rceil = \frac{n_1}{2} - 1$ (as n_1 is even). That is, G_1 is the intersection of at most $\frac{n_1}{2} - 1$ interval graphs, say $I_1', I_2', \ldots, I'_{\frac{n_1}{2}-1}$, by Theorem 2.8.

Let h_i be a function that maps each $v \in V(I_i')$, $1 \leq i \leq \frac{n_1}{2} - 1$, to a closed interval on the real line such that I_i' is the intersection graph of the family of intervals $\{h_i(v) : v \in V(I_i')\}$. Now, let L_i' and R_i', $1 \leq i \leq \frac{n_1}{2} - 1$, be numbers on the real line such that $L_i' \leq x$, for all $x \in \bigcup_{v \in V(I_i')}(h_i(v))$ and $R_i' \geq y$, for all $y \in \bigcup_{v \in V(I_i')}(h_i(v))$. To construct I_i, $1 \leq i \leq \frac{n_1}{2} - 1$, map each $v \in V(G)$ to a closed interval $f_i(v)$ on the real line as follows.

$$\begin{aligned} f_i(v) &= h_i(v) \qquad if\ v \in V(I'_i) = V(G) - \{x, y\}. \\ &= [L'_i, R'_i] \quad otherwise. \end{aligned}$$

CLAIM 7. *For each I_i, $1 \le i \le \frac{n_1}{2} - 1$, $E(G) \subseteq E(I_i)$.*

Proof. Since $f_i(x) = f_i(y) = [L'_i, R'_i]$ it is easy to see that in I_i, x and y are adjacent to each v, $v \in V(I_i) - \{x, y\}$, by the definition of L'_i and R'_i. As $I'_1, I'_2, \ldots, I'_{\frac{n_1}{2}-1}$ is an interval graph representation of G_1, by Theorem 2.8, $E(G_1) \subseteq E(I'_i)$, for each $1 \le i \le \frac{n_1}{2} - 1$. But in I_i, $f_i(v) = h_i(v)$, for all $v \in V(I'_i)$. So, $E(G_1) \subseteq E(I_i)$, $1 \le i \le \frac{n_1}{2} - 1$. Hence for each I_i, $1 \le i \le \frac{n_1}{2} - 1$, $E(G) \subseteq E(I_i)$. □

Construction of $I_{\frac{n_1}{2}}$:

Let $V_1 = \{v_1, v_2, \ldots, v_{n_1}\}$. Suppose without loss of generality that $x = v_1$ and $y = v_{n_1}$. To construct $I_{\frac{n_1}{2}}$, we map each $v \in V(G)$ to an interval $f_{\frac{n_1}{2}}(v)$ as follows.

$$\begin{aligned} f_{\frac{n_1}{2}}(v) &= [2i-1, 2i] && \text{if } \ v \in V_1 \ \text{ and } \ v = v_i \\ &= [1, 2n_1] && \text{if } v \in V_2 \text{ and } \ v \in N_x \cap N_y. \\ &= [1, 2n_1 - 2] && \text{if } v \in V_2 \text{ and } \ v \in N_x - N_y. \\ &= [3, 2n_1] && \text{if } v \in V_2 \text{ and } \ v \in N_y - N_x. \\ &= [3, 2n_1 - 2] && \text{if } \ v \in V_2 - (N_x \cup N_y). \end{aligned}$$

CLAIM 8. *$E(G) \subseteq E(I_{\frac{n_1}{2}})$.*

Proof. In $I_{\frac{n_1}{2}}$, for each $v \in V_2$, the point $n_1 \in f_{\frac{n_1}{2}}(v)$. So, V_2 induces a clique in $I_{\frac{n_1}{2}}$. Also for each $v \in N_G(x)$, $1 \in f_{\frac{n_1}{2}}(v)$ and for each $v \in N_G(y)$, $2n_1 \in f_{\frac{n_1}{2}}(v)$. So, $N_G(x) \subseteq N_{I_{\frac{n_1}{2}}}(x)$ and $N_G(y) \subseteq N_{I_{\frac{n_1}{2}}}(y)$. For $v_j \in V_1 - \{x, y\}$, we have $2 \le j \le n_1 - 1$, and thus we have $3 \le 2j - 1 \le 2n_1 - 2$. So, $2j - 1 \in f_{\frac{n_1}{2}}(v)$, for all $v \in V_2$. It is easy to see that $(v_i, v) \in E(I_{\frac{n_1}{2}})$ for all pairs (v_i, v) where $v_i \in V_1 - \{x, y\}$ and $v \in V_2$. Hence $E(G) \subseteq E(I_{\frac{n_1}{2}})$. □

The following Lemma follows from Claims 7 and 8.

LEMMA 3.18. *For each interval graph I_i, $1 \le i \le \frac{n_1}{2}$, $E(G) \subseteq E(I_i)$.*

LEMMA 3.19. *For any $(p,q) \notin E(G)$, there exists some i, $1 \le i \le \frac{n_1}{2}$, such that $(p,q) \notin E(I_i)$.*

Proof. Suppose $(p,q) \notin E(G)$.

Case 1: $\{p,q\} \subseteq V_1$.

Suppose $p = v_i$ and $q = v_j$, where $i \ne j$. In this case it is easy to see that in $I_{\frac{n_1}{2}}$, $f_{\frac{n_1}{2}}(p) \cap f_{\frac{n_1}{2}}(q) = \emptyset$. So, p is non-adjacent to q in $I_{\frac{n_1}{2}}$.

Case 2: $\{p,q\} \subseteq V_2 \cup V_1'$.

If both p and q belong to $V_2 \cup V_1'$, then it is easy to see that $p, q \in G_1$ and in view of Case 1, $(p,q) \notin E(G_1)$. As $I_1', I_2', \ldots, I'_{\frac{n_1}{2}-1}$ is an interval graph representation of G_1, by Theorem 2.8, $(p,q) \notin E(I_i')$, for some i, $1 \le i \le \frac{n_1}{2} - 1$, say I_k'. Recalling that in I_k, $f_k(v) = g_k(v)$ for all $v \in V(I_k')$ p and q are non-adjacent in I_k also.

Case 3: $p \in \{x, y\}$ and $q \in V_2$.

Let $p = x$. Now, in $I_{\frac{n_1}{2}}$, $f_{\frac{n_1}{2}}(x) = [1,2]$ and as q is not a neighbor of x in G, either $f_{\frac{n_1}{2}}(q) = [3, 2n_1 - 2]$ or $[3, 2n_1]$. In both the cases, $f_{\frac{n_1}{2}}(p) \cap f_{\frac{n_1}{2}}(q) = \emptyset$. So, p and q are non-adjacent in $I_{\frac{n_1}{2}}$.

Similarly, if $p = y$, then in $I_{\frac{n_1}{2}}$, $f_{\frac{n_1}{2}}(y) = [2n_1 - 1, 2n_1]$ and as q is not a neighbor of y in G, either $f_{\frac{n_1}{2}}(q) = [1, 2n_1 - 2]$ or $[3, 2n_1 - 2]$. In both the cases, $f_{\frac{n_1}{2}}(p) \cap f_{\frac{n_1}{2}}(q) = \emptyset$. So, p and q are non-adjacent in $I_{\frac{n_1}{2}}$. □

By Lemmas 3.18 and 3.19, we get $E(G) = E(I_1) \cap E(I_2) \cap \cdots \cap E(I_{\frac{n_1}{2}})$. Thus by Theorem 2.8, we have the following.

THEOREM 3.20. *For a bipartite graph $G = (V_1 \cup V_2, E)$, $box(G) \le \min\left\{\left\lceil \frac{n_1}{2} \right\rceil, \left\lceil \frac{n_2}{2} \right\rceil\right\}$, where $|V_1| = n_1$ and $|V_2| = n_2$.*

3.5.1 Tightness result

In this section we show that the bound given in Theorem 3.20 is tight. Consider a complete bipartite graph $G = (V_1 \cup V_2, E)$ and remove a perfect matching from that. Let $G' = (V_1 \cup V_2, E')$ be the resulting graph. It is easy to see that, $|V_1| = |V_2| = \frac{n}{2}$ (as G has a perfect matching). Next we show that $box(G') = \lceil \frac{n}{4} \rceil$.

CLAIM 9. $box(G') \geq \lceil \frac{n}{4} \rceil$.

Proof. If possible suppose $box(G') \leq \lceil \frac{n}{4} \rceil - 1$. Let n be divisible by 4. So, $\lceil \frac{n}{4} \rceil - 1 = \frac{n}{4} - 1$, By Theorem 2.8, G' is the intersection of at most $\frac{n}{4} - 1$ interval graphs. Recall that, $\frac{n}{2}$ edges (size of a perfect matching of G) are missing in G'. Since there are total $\frac{n}{2}$ missing edges, at least one interval graph, say I_k, $1 \leq k \leq \frac{n}{4} - 1$, will be such that at least three edges, say aa', bb', and cc', will be absent in I_k.

Let $\{a, b, c\} \subseteq V_1$ and $\{a', b', c'\} \subseteq V_2$. So, $\{a, b', c, a', b, c'\}$ (=C, say) is a cycle of length six in I_k. By Lemma 2.2, C cannot be an induced cycle in I_k, otherwise I_k will not be an interval graph. Consider the following cases:

Case 1. Either $\{a, b, c\}$ or $\{a', b', c'\}$ induces a complete graph in I_k.

Without loss of generality let $\{a, b, c\}$ induce a complete graph in I_k. If $I_k[\{a', b', c'\}]$ has no edge, then by Lemma 2.11, $I_k = S_2$ is not an interval graph. So, suppose $I_k[\{a', b', c'\}]$ contains at least one edge. Without loss of generality suppose $a'b' \in E(I_k)$. Now, it is easy to see that $\{a, b, a', b'\}$ induces a cycle of length four in I_k. This is a contradiction.

Case 2. Both $I_k[\{a, b, c\}]$ and $I_k[\{a', b', c'\}]$ are not complete.

Suppose $I_k[\{a, b, c\}]$ or $I_k[\{a', b', c'\}]$ has two edges. Without loss of generality suppose $I_k[\{a, b, c\}]$ has two edges ab and bc. Now, $\{a, b', c, b\}$ induces a cycle of length four in I_k. This is a contradiction. So, $I_k[\{a, b, c\}]$ and $I_k[\{a', b', c'\}]$ contains exactly one edge each. Without loss of generality suppose $ab \in E(I_k)$. If $a'b' \in E(I_k)$, then $\{a, b, a', b'\}$ induces a cycle of length four in I_k. This is a contradiction. Now, either $a'c' \in E(I_k)$ or $b'c' \in E(I_k)$. If $a'c' \in E(I_k)$, then it is easy to see that $\{a, b', c, a', c'\}$ induces a cycle of length five in I_k, which is a contradiction. Similarly if $b'c' \in E(I_k)$, then also there will be an induced cycle of length five in I_k, which is a contradiction. So, this case is also not possible.

Hence, I_k cannot contain three missing edges. This is a contradiction.

Next suppose n is not divisible by 4. Now, $\lceil \frac{n}{4} \rceil - 1 = \lfloor \frac{n}{4} \rfloor + 1 - 1 = \lfloor \frac{n}{4} \rfloor$. Using similar arguments as in the case when n is divisible by 4, we will get a contradiction in this case also.

Hence, $box(G') \geq \lceil \frac{n}{4} \rceil$. □

By Theorem 3.20, we have $box(G') \leq \lceil \frac{n}{4} \rceil$. Hence $box(G') = \lceil \frac{n}{4} \rceil$. So, for such graphs the bound given for bipartite graphs is tight.

3.6 Boxicity and Chromatic Number

We know that $box(G) \leq \lfloor \frac{n}{2} \rfloor$, where n is the number of vertices of G (Theorem 3.11). Let $box(G) = \frac{n}{2} - s$, for some $s \geq 0$. Note that, if n is odd, then s is not an integer. In the following theorem, we show that when s is small for a graph G, the chromatic number of G has to be very high.

THEOREM 3.21. *([2]) If $box(G) = \frac{n}{2} - s$, then $\chi(G) \geq \frac{n}{2s+2}$.*

Proof. Let $box(G) = \frac{n}{2} - s$. By Theorem 3.16, $box(G) = \frac{n}{2} - s \leq \lfloor \frac{t}{2} \rfloor + 1 \leq \frac{t}{2} + 1$, where t is the cardinality of a minimum vertex cover of G. So, $t \geq n - 2s - 2$. It is easy to see that if α is the independence number of G, then $\chi(G) \geq \frac{n}{\alpha}$. But $\alpha = n - t$. So,

$$\begin{aligned}
\chi(G) &\geq \frac{n}{n-t} \\
&\geq \frac{n}{n-(n-2s-2)} \\
&= \frac{n}{2s+2}
\end{aligned}$$

□

The lower bound for $\chi(G)$ given in Theorem 3.21 is tight for the graph $T(n, \frac{n}{2})$. From Lemma 3.2, we know that $box(T(n, \frac{n}{2})) = \frac{n}{2}$ (recall that n is even in $T(n, \frac{n}{2})$). Thus, $s = 0$ for $T(n, \frac{n}{2})$. Substituting $s = 0$ in the inequality given in Theorem 3.21, we get $\chi(G) \geq \frac{n}{2}$. But $\chi(G) = \frac{n}{2}$ for $T(n, \frac{n}{2})$.

Chapter 4

Cubicity

4.1 Introduction

In Chapter 2.5 we have seen an example of how to prove a lower bound on the cubicity of a graph ($K_{1,r}$). In Section 4.2, we prove an upper bound on the cubicity of a graph in terms of its minimum vertex cover: $cub(G) \leq t + \lceil \log\ (n-t) \rceil - 1$, where t is the cardinality of a minimum vertex cover.

4.2 Cubicity and Vertex Cover

In this section, we give a tight upper bound for cubicity of a graph G in terms of the cardinality of its minimum vertex cover. In particular we show that $cub(G) \leq t + \lceil \log\ (n-t) \rceil - 1$, where $|MVC| = t$ and n is the number of vertices of G. Let $MVC = \{v_1, v_2, \ldots, v_t\}$. Clearly $A = V - MVC$ is an independent set in G. Let $A = \{w_0, w_1, \ldots, w_{\alpha-1}\}$, where $|A| = n - t = \alpha$. Next, we construct $t + \lceil \log\ (n-t) \rceil - 1$, unit interval super graphs of G, say $U_1, U_2, \ldots, U_{t+\lceil \log\ (n-t) \rceil - 1}$, as follows.

Construction of U_i for $1 \leq i \leq t-1$:

Let $MVC' = MVC - \{v_t\}$. So, $|MVC'| = t - 1$. For each $v_i \in MVC'$, $1 \leq i \leq t-1$, we construct a unit interval graph U_i. To construct U_i, map each $x \in G$ to a unit interval

$f_i(x)$ as follows.

$$\begin{aligned} f_i(x) &= [0,1] && if\ x = v_i \\ &= [1,2] && if\ x \in N_G(v_i). \\ &= [2,3] && if\ x \in V(G) - (N_G(v_i) \cup \{v_i\}). \end{aligned}$$

CLAIM 10. *For each unit interval graph U_i, $1 \leq i \leq t-1$, $E(G) \subseteq E(U_i)$.*

Proof. It is easy to see that for all $x \in N_G(v_i) \cup \{v_i\}$, $1 \in f_i(x)$. So, $N_G(v_i) \cup \{v_i\}$ induces a clique in U_i. Also, for all $x \in V(G) - \{v_i\}$, $2 \in f_i(x)$. That is, $V(G) - \{v_i\}$ induces a clique in U_i. So, we infer that $E(G) \subseteq E(U_i)$, for each i, $1 \leq i \leq t-1$. □

Construction of U_{t+j} for $0 \leq j \leq \lceil \log(n-t) \rceil - 1$:

Recall that $MVC = \{v_1, v_2, \ldots, v_t\}$ and $A = \{w_0, w_1, \ldots, w_{\alpha-1}\}$. It is easy to see that v_t is adjacent to at least one vertex of A since MVC is a minimum vertex cover of G. Without loss of generality suppose $(v_t, w_0) \in E(G)$. For each j, $0 \leq j \leq \lceil \log\ (n-t) \rceil - 1$, we define a function $b_j : A \rightarrow \{0, 1\}$ as follows:

$$\begin{aligned} b_j(w_k) &= 0 && if\ the\ (j+1)-th\ least\ significant\ bit\ of\ k\ is\ 0 \\ &= 1 && otherwise. \end{aligned}$$

To construct U_{t+j}, $0 \leq j \leq \lceil \log\ (n-t) \rceil - 1$, we map each $x \in V(G)$ to a unit interval as follows.

$$\begin{aligned} f_{t+j}(x) &= [0.5, 1.5] && if\ x = v_t. \\ &= [1,2] && if\ x \in MVC'. \\ &= [0,1] && if\ x = w_0. \\ &= [0,1] && if\ x \in A - \{w_0\}\ and\ b_j(x) = b_j(w_0). \\ &= [1.5, 2.5] && if\ x \in A - \{w_0\}\ and\ b_j(x) \neq b_j(w_0)\ and\ xv_t \in E(G). \\ &= [2,3] && if\ x \in A - \{w_0\}\ and\ b_j(x) \neq b_j(w_0)\ and\ xv_t \notin E(G). \end{aligned}$$

Claim 11. *For each unit interval graph U_{t+j}, $0 \le j \le \lceil \log\,(n-t) \rceil - 1$, $E(G) \subseteq E(U_{t+j})$.*

Proof. It is easy to see that, for all $x \in MVC$, $1 \in f_{t+j}(x)$. So, MVC induces a clique in U_{t+j}. Also, for all $y \in N_G(v_t)$, either $1 \in f_{t+j}(y)$ or $1.5 \in f_{t+j}(y)$. As $f_{t+j}(v_t) = [0.5, 1.5]$, $f_{t+j}(v_t) \cap f_{t+j}(y) \neq \emptyset$, for all $y \in N_G(v_t)$. So, $N_G(v_t) \subseteq N_{U_{t+j}}(v_t)$. Let $w_i \in A$. Now, either $f_{t+j}(w_i) = [0, 1]$ or $[1.5, 2.5]$ or $[2, 3]$. In all the cases, it is easy to see that $f_{t+j}(w_i) \cap f_{t+j}(v) \neq \emptyset$, for all $v \in MVC'$ since $f_{t+j}(v) = [1, 2]$. That is, for each $w_i \in A$, $w_i v \in E(U_{t+j})$, for all $v \in MVC'$. Hence for each j, $0 \le j \le \lceil \log\,(n-t) \rceil - 1$, $E(G) \subseteq E(U_{t+j})$. □

The following Lemma follows from Claims 10 and 11.

Lemma 4.1. *For each unit interval graph U_i, $1 \le i \le t + \lceil \log\,(n-t) \rceil - 1$, $E(G) \subseteq E(U_i)$.*

Lemma 4.2. *For any $(x, y) \notin E(G)$, there exists some i, $1 \le i \le t + \lceil \log\,(n-t) \rceil - 1$, such that $(x, y) \notin E(U_i)$.*

Proof. Suppose $(x, y) \notin E(G)$.

Case 1: $\{x, y\} \subseteq MVC$.

It is easy to see that either x or y, say x, will be present in MVC'. Let $x = v_i$, for some i, $1 \le i \le t-1$. Now, in U_i, as $y \notin N_G(v_i)$, $f_i(x) = [0, 1]$ and $f_i(y) = [2, 3]$. So, $f_i(x) \cap f_i(y) = \emptyset$. Hence, x is non-adjacent to y in U_i.

Case 2: $x \in MVC$ and $y \in A$.

First suppose $x \in MVC'$. Let $x = v_i$, for some i, $1 \le i \le t-1$. Now, in U_i, as $y \notin N_G(v_i)$, $f_i(x) = [0, 1]$ and $f_i(y) = [2, 3]$. Hence, x is non-adjacent to y in U_i.

Next suppose $x = v_t$. It is easy to see that $y \neq w_0$, as $(w_0, v_t) \in E(G)$ by assumption. Let $y = w_s$, for some s, $1 \le s \le \alpha - 1$. Since $s > 0$, clearly there exists a l, $0 \le l \le \lceil \log\,(n-t) \rceil - 1$, such that $b_l(w_s) \neq b_l(w_0)$. Now, in U_{t+l}, $f_{t+l}(w_s) = [2, 3]$. But $f_{t+l}(v_t) = [0.5, 1.5]$. As $f_{t+l}(v_t) \cap f_{t+l}(w_s) = \emptyset$, x and y are non-adjacent in U_{t+l}.

Case 3: $\{x, y\} \subseteq A$.

Let $x = w_r$ and $y = w_s$, $0 \le r, s \le \alpha - 1$. Since $r \neq s$, there exists a j, $0 \le j \le \lceil \log\,(n-t) \rceil - 1$, such that $b_j(w_r) \neq b_j(w_s)$. As $b_j(w_0)$ is either 0 or 1, $b_j(w_0)$ is different from either $b_j(w_r)$ or $b_j(s)$. Without loss of generality suppose $b_j(w_0) \neq b_j(w_s)$. So, $b_j(w_0) = b_j(w_r)$ as $b_j(w_r) \neq b_j(w_s)$. Now, in U_{t+j}, $f_{t+j}(w_r) = [0, 1]$ and $f_{t+j}(w_s) = [1.5, 2.5]$ or $[2, 3]$. In both the cases $f_{t+j}(w_r) \cap f_{t+j}(w_s) = \emptyset$. Hence $x = w_r$ and $y = w_s$ are non-adjacent in U_{t+j}, $0 \leq j \leq \lceil \log\ (n-t) \rceil - 1$. □

By Lemmas 4.1 and 4.2, we get
$E(G) = E(U_1) \cap E(U_2) \cap \cdots \cap E(U_{t+\lceil \log(n-t) \rceil - 1})$. Thus by Theorem 2.8, we have the following:

THEOREM 4.3. *([2]) For a graph G, $cub(G) \leq t + \lceil \log\ (n-t) \rceil - 1$, where $|MVC| = t$ and n is the number of vertices of G.*

It is easy to see that for a bipartite graph G, $t \leq \frac{n}{2}$, where t is the cardinality of a minimum vertex cover of G. Applying the bound given above, we have the following result.

COROLLARY 4.4. *For a bipartite graph G, $cub(G) \leq \frac{n}{2} + \lceil \log n \rceil - 1$.*

For a graph G, it is known that $cub(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor$ [22]. Note that for bipartite graphs Corollary 4.4 gives a better upper bound.

4.2.1 Tightness result

In this section we show that the upper bound given for cubicity in Theorem 4.3 is tight for $K_{1,n-1}$. It is easy to see that $|MVC| = 1$ for $K_{1,n-1}$. Substituting $t = 1$ in the inequality of Theorem 4.3 gives us the inequality $cub(K_{1,n-1}) \leq \lceil \log(n-1) \rceil$. But, by Theorem 2.15, we know that $cub(K_{1,n-1}) = \lceil \log(n-1) \rceil$. So, the upper bound for cubicity given in Theorem 4.3 is tight for $K_{1,n-1}$.

Chapter 5

Conclusions and Future Work

In this thesis, we saw the known result that for any graph G, $box(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ [22] and improved this result by proving that $box(G) \leq \left\lfloor \frac{t}{2} \right\rfloor + 1$. As shown in Corollary 3.17, this result improves upon the previous result in [22]. This bound is tight.

We also saw the known result that for any graph G, $cub(G) \leq \left\lfloor \frac{2}{3}n \right\rfloor$ [22] and gave the new bound $cub(G) \leq t + \lceil log_2(n-t) \rceil - 1$. This result is not always better than the result in [22], but is tight nonetheless.

5.1 Directions for Future Work

It is NP-hard to compute the boxicity of a general graph [9]. However, it would be nice if one can characterize or find some non-trivial properties of the graphs with boxicity k. [18] gives a characterization of the graphs with boxicity less than or equal to 2. Finding the boxicity and cubicity of some special classes of graphs would also be useful.

Let P be a set of n points. We associate with each point $p \in P$ an open ball B_p, which is centered at p and has radius equal to the distance between p and its nearest neighbour.

Definition 5.1. *A sphere-of-influence graph is a graph $G(V, E)$ with $V(G) = P$ and for $\{p_1, p_2\} \subseteq V(G)$, $(p_1, p_2) \in E(G) \Leftrightarrow B_{p_1} \cap B_{p_2} \neq \emptyset$.*

The points could by lying in $\mathbb{R}^d$ and the distance between them found using the Euclidean metric. In [13], the metric induced from the sup-norm (L_∞-norm) is considered.

Definition 5.2. *Given a graph G, the minimum dimension d for which we can map the vertices of G to points in $\mathbb{R}^d$ such that the sphere-of-influence graph so realized is isomorphic to G, is called the Sphere-of-Influence Graph dimension of G denoted by $SIG(G)$.*

In [13], it is shown that in the L_∞-norm, $SIG(G) \leq t + \lceil \log(n-t) \rceil - 1$ where $t = |MVC|$.

Note that this inequality is same as the inequality we proved for cubicity in Theorem 4.3. However, cubicity and SIG dimension are not the same. We would like to explore the SIG dimension further to see if either of the dimensions helps in understanding the other dimension better.

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