# Reconstructing Sets From Interpoint Distances

Paper presented in 1995 by


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The Problem

- Which point sets realize a given distance multiset?

# Reconstructing Sets From Interpoint Distances

## Outline Of The Presentation

- O($n^n \log n$) backtracking algorithm for Beltway Problem.
- Bounds and other properties of Homometric Sets.
  - Turnpike problem
  - Beltway Problem
- Constructing Homometric Beltway Sets with Singer Difference Sets.
- Bounds on Homometric Sets for the $d$ – dimensional problem. (If time permits)
- Backtracking Algorithm for reconstruction in $d$ – dimensional space.(If time permits)

# Beltway Algorithm
## Some Identities

- Distances are taken *mod L*, indices *mod n.*
- Fix a direction for the distances, say Clockwise.
- $\binom{n}{2}$ L = $\sum_{0 \le i,\, j < n} d_{ij}$ So, find L.
- k L = $\sum_{0 \le i < n} d_{i(i+k)}$
- $d_{ij} + d_{jk} = d_{ik}$ Triangle equality. Important.
- $d_{ij} + d_{ji} =$ L Complement Distance Identity.

# Beltway Problem
## The Algorithm

- Given n(n-1) distances.
- Construct an (n-1) x n table.
- k*th* row : $d_{i(i+n-k)}$ i = 0,1, ... ,n-1 ; k = 1,2, ... ,n-1
- $d_{ij} \geq d_{i(j-1)}$ Entry in table below a given entry.

  $d_{ij} \geq d_{(i-1)j}$ Entry in table below and to the right of a given entry.
- Fill in the distances largest first. Only need to choose which column to place the next distance in. Why?

# Beltway Algorithm

## Analyzing the Algorithm

- After choosing which column to place the next distance in, the identity

$$d_{ij} + d_{jk} = d_{ik}$$

allows us to fill in various distances.

- Total $(n-1)^2$ fill-ins.

Each fill-in takes O(log*n) time.*

- Decisions by the backtrack algorithm : n-2 (First choice – arbitrary)

Each decision has at most n choices.

Each choice causes at most $n^2$ fill-ins.

- Eventual time taken : O( $n^n$ log*n*).
- Space taken : merely O( $n^2$ ).

# Homometric Sets
## Definitions and Terminology

- Two noncongruent n-point sets are homometric if the multisets of $\binom{n}{2}$ distances they determine are the same.
- Equivalently, a given distance multiset may realize more than one homometric sets of distances.
- $H_d(n)$ denotes the maximum possible number of noncongruent, homometric n-point sets which can exist in $\mathbb{R}^d$ .
- $S_d(n)$ denotes the maximum possible number of such sets for which all points lie on sphere $S^d$ .
- $H_d*(n)$ same as $H_d(n)$ except that overlapping points are allowed. $H_d(n) \le H_d*(n)$ .
- $S_d*(n)$ is defined similarly. $S_d(n) \le S_d*(n)$.
- $S_d(n) \le H_{d+1}(n)$

# Homometric Sets - Turnpike

## Lower Bounds

- For $n \leq 5$, $H_1(n)$ = 1.
  - n = 2, 3 : Trivial.
  - n=4 : Not so tough.
  - n=5 : x1 = 0, x5 = d.

    9 distances remain. 3 pairs sum upto d.

    3 distances remain – {a,b,c}.

    Case -1: $a+c=d$ ; b determines a,b,c.

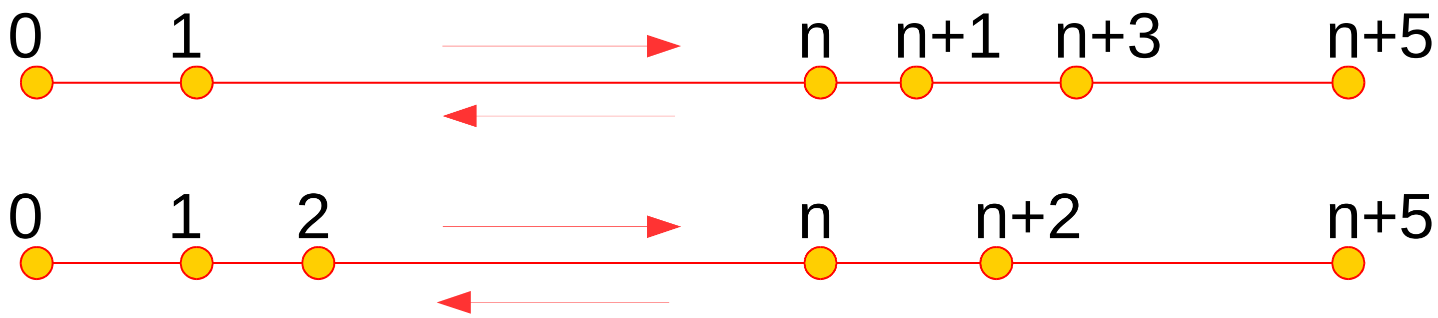
    Case -2: $a+c \neq d$ ; 3 pairs determine a,b,c.

# Homometric Sets - Turnpike

## Lower Bounds

- $n \geq 6$ , $H_1(n) \geq 2$
- Set-1: $X = \{n+1, n+3\} \cup S \cup T$
- Set-2: $Y = \{2, n+2\} \cup S \cup T$
- S = { i | $5 \leq i \leq n-2$ }
- T = {0,1,n,n+5}
- Claim : X and Y are homometric.
- X : n+1,n,1,4, n+3,n+2,3,2,2.
- Y : 2,1,n-2,n+3, n+2,n+1,2,3,n.
- Unbalanced: X – 4, Y – n-2. Remember.
- Now, we take distances with elements of set S.

# Homometric Sets - Turnpike
## Lower Bounds – Proving the Claim




X : **5+k**,4+k,n-5-k,n-4-k,n-2-k,n-k, **n-2-k**,n-3-k,2+k,3+k,5+k,7+k

Y : n-2-k,**n-3-k**,n-4-k,2+k,4+k,7+k, 5+k,**4+k**,3+k,n-5-k,n-3-k,n-k

Remaining:

X : 5+k, n-2-k Y : 4+k,n-3-k

X : n-2,n-2 Y : 4,4 Cancell off. Why?

# Homometric Sets - Turnpike
## Lower Bounds

- For an infinite number of values of n,

$$\frac{1}{2}n^{\alpha} \leq H_1(n) \qquad \alpha = \frac{\ln(8)}{\ln(13)} \approx 0.8107$$

- Known that $H_1(13) \geq 4$ , $H_1(ab) \geq 2H_1(a)H_1(b)$
- So, $H_1(13^k) \geq 2^{3k-1}$ = x (say).
- Now, ln(n)=k.ln(13)
- $\frac{\ln(x)}{\ln(13)} = k.\frac{\ln(8)}{\ln(13)}$ $\qquad \ln(x) = \ln(n^{\frac{\ln(8)}{\ln(13)}})$
- In general, if $H_1(\alpha) \geq r$ , for $n = \alpha^k$ : $\quad H_1(n) \geq \frac{1}{2}n^{\frac{\ln(2r)}{\ln(\alpha)}}$
- Open Problem : Improve the bound

  For eg. demonstrate that for some $n \leq 30$ , $H_1(n) \geq 8$

# Homometric sets - Beltway
## Trivial Lower Bounds

- $S_1(n)=1$ for $n\leq 3$
- $S_1(n)\geq 2$ for $n\geq 4$
- X : {0,t,1+t,2,4, ... ,2(n-3)}
- Y : {0,t,2,4, ... ,2(n-3),2n-5+t} mod 2(n-2)

# Homometric Sets - Beltway
## Singer Difference Sets

- Singer Difference Sets are *n* – element subsets of $\mathbb{Z}_M$ where
  M = $q^2+q+1$ and q is a prime; n = q + 1
  such that the n(n-1) differences they determine are exactly all
  the non-zero elements of $\mathbb{Z}_M$.
- Singer showed that for each prime q, there always exists at least one such example.
- Multiplying a Singer Set by any element r of $\mathbb{Z}_M$ preserves the distance set. If r | q, then this may result to translation of the set.
- If $q^2+q+1$ is a prime, gcd(M,r) =1, q and -1 generate dihedral group $D_3$
- If no further multiplicative symmetries exist, then it would have (M-1)/6 or q(q+1)/6 equivalance classes of size 6.

# Homometric sets - Beltway
## Singer Difference Sets

- Hardy – Littlewood Conjecture :

  There are infinite number of primes q such that $q^2+q+1$

  is also a prime.

- Our Conjecture :

  Singer Set construction generates at least q(q+1)/6 homometric (q+1) – point beltway sets where q and $q^2+q+1$ are both primes. This would follow if we can show that at least one Singer set with no nontrivial symmetries exists.

# Homometric Sets
## *d* – dimensional problem

- d points generate a (d-1) – simplex which we shall call the central simplex.
- Claim :

  (n-d)d + $\binom{d}{2}$ line segments form a rigid structure and the point set is completely defined.
- The (n-d) points are assigned +/- signs depending on which side of the hyperplane they lie.
- No. of ways < $\binom{n}{2}^{(n-d)d+\binom{d}{2}} * 2^{n-d}$
- We can remove a factor of d!(n-d)! due to automorphisms.

# The Algorithm
## *d* – dimensional problem

- Construct the central simplex using d/2 of the largest distances.in the x1=0 hyperplane.
- For each central simplex, consider the addition of one point at a time by backtracking.
- A point is added by selecting d distances to the central simplex and its sign. Backtracking occurs if
  - A valid d-simplex is not formed
  - The distances to currently embedded points are not valid.
  - The d-tuple of distances is lexicographically larger than some previous d-tuple. This eliminates (n-d)! automorphisms
- Embedding a point – O($d^3$)
- Finding n distances – O($n \log n$).
- O( $n^{(2d-1)n} e^n$ )

# Thank You