EXERCISES - SET II

1. Let $A \in \mathbb{R}^{4\times 5}$ be as given below:

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & -1 & 3 \\ 2 & 4 & 3 & -1 & 4 \\ 3 & 6 & 1 & 2 & -1 \end{pmatrix}$$

Answer the following: Find a basis for

(a) $\mathcal{R}_{A}$

(b) $\mathcal{R}_{A^T}$

(c) $\mathcal{N}_{A}$

2. Let $A \in \mathbb{R}^{5\times 4}$ be such that the RRE form of A^T is given by

$$(A^T)_R = \begin{pmatrix} 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer the following :

(a) Find a basis for $\mathcal{R}_{A}$

(b) Determine which of the following vectors are in $\mathcal{R}_{A}$:

$$b = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

3. The RRE form of a matrix $A \in \mathbb{R}^{4\times 5}$ is

$$A_R = \begin{pmatrix} 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Determine whether the following statements are TRUE or FALSE:

(a) Dimension of $\mathcal{R}_{A}$ is 2

(b) The first two columns of A together form a basis for $\mathcal{R}_{A}$

(c) The first three columns of A together form a basis for $\mathcal{R}_{A}$

(d) The first and fourth columns of A ogether form a basis for $\mathcal{R}_{A}$

(e) The first, fourth and fifth columns of A together form a basis for $\mathcal{R}_{A}$

4. Consider the matrix $A \in \mathbb{R}^{4\times 3}$ given below:

$$A = \begin{pmatrix} 1 & 4 & 2 & -1 & 1 \\ -2 & -8 & -3 & 1 & 1 \\ 2 & 8 & 2 & 1 & -5 \\ -1 & -4 & -1 & 2 & 0 \end{pmatrix}$$

Answer the following:

(a) Find the RRE form A_R of A

(b) Find the general solution of the Homogeneous System $Ax = \theta_4$

(c) Find the condition(s) that $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \in \mathbb{R}^4$ has to satisfy in order that the system $Ax = b$ is consistent

(d) When $b \in \mathbb{R}^4$ satisfies the consistency condition(s) find the general solution of the system $Ax = b$

5. In the vector space $\mathbb{R}^4$ determine whether the following sets of vectors are linearly independent:

(a) $\mathcal{S} : u_1 = \begin{pmatrix} 1 \\ -3 \\ 0 \\ 2 \end{pmatrix}, \; u_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; u_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 3 \end{pmatrix}$

(b) $\mathcal{S} : u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \; u_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \; u_3 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \; u_4 = \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \end{pmatrix}$

6. For the sets $\mathcal{S}$ in the above example find a basis for $\mathcal{L}[\mathcal{S}]$ and the dimension of $\mathcal{L}[\mathcal{S}]$

7. Let $\mathcal{S} = u,\ v,\ w$ be a linearly independent set in $\mathbb{R}^3$. Determine whether the set $\mathcal{S}' = u',\ v',\ w'$ is linearly independent, where
$$u' = u + v,\ v' = v + w,\ w' = w + u$$

8. For what values of $\alpha \in \mathbb{R}$ are the following vectors in $\mathbb{R}^3$ linearly independent?
$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix},\ v_3 = \begin{pmatrix} 5 \\ 3 \\ \alpha \end{pmatrix}$$

9. If u, v, w are linearly independent vectors in a $\mathcal{V}$, for what value(s) of k are the vectors $u',\ v',\ w'$ linearly independent where
$$u' = v - u,\ v' = kw - v,\ w' = u - w$$

10. Let $u,\ v,\ w,\ z$ be vectors in $\mathbb{R}^n$ such that u, v, w is a linearly dependent set and v, w, z is a linearly independent set. Prove the following:
    (a) u is a linear combination of v, w **and**
    (b) z is NOT a linear combination of u, v, w

11. Consider the set S in $\mathbb{R}^4$ defined below:
$$S : v_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 2 \\ 8 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v_5 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 2 \end{pmatrix}$$
Find a linearly independent subset $\tilde{S}$ of S such that $\tilde{S}$ is a basis for $\mathcal{L}[S]$.

12. Find a subset $\tilde{\mathcal{C}}$, of the set of the column vectors of the following matrix $A \in \mathbb{R}^{4\times 5}$, such that $\tilde{\mathcal{C}}$ is a basis for $\mathcal{R}_{A}$:
$$A = \begin{pmatrix} 1 & 2 & 4 & 8 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 2 & 0 \end{pmatrix}$$

13. Show that the following set of vectors form a basis for $\mathbb{R}^3$:

$$\mathcal{B} : v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

14. Show that the following set of matrices form a basis for $\mathbb{R}^{2\times 2}$:

$$\mathcal{B} : A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \ A_4 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

15. Consider the subspace $\mathcal{W}$ of $\mathbb{R}^4$ defined as follows:

$$\mathcal{W} = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ 2\alpha + \beta \\ 0 \end{pmatrix} : \alpha, \ \beta \in \mathbb{R} \right\}$$

Answer the following:

(a) Find a basis for $\mathcal{W}$ and hence the dimension of $\mathcal{W}$

(b) Extend the basis for $\mathcal{W}$ obtained above to a basis for $\mathbb{R}^4$

16. Consider the vector space $\mathcal{V} = \mathbb{C}^{2\times 2}$ **over the field $\mathbb{R}$ of real numbers**. Let $\mathcal{W}$ be the subspace of this defined as follows:

$$\mathcal{W} = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} : \alpha, \ \beta, \ \gamma \in \mathbb{R} \right\}$$

Find a basis for $\mathcal{W}$ and hence the dimension of $\mathcal{W}$.

17. Let $\mathcal{V}$ and $\mathcal{U}$ be any two vector spaces over a field $\mathbb{F}$. The addition and scalar multiplication in these two spaces are denoted respectively by, $x \oplus_{\mathcal{V}} y$, $\alpha \odot_{\mathcal{V}} x$, and $x \oplus_{\mathcal{U}} y$ and $\alpha \odot_{\mathcal{U}} x$. Let $\mathcal{W}$ be defined as follows:

$$\mathcal{W} = \{(v, u) : v \in \mathcal{V}, \ u \in \mathcal{U}\}$$

Define addition on $\mathcal{W}$ as follows:

$$\begin{aligned} x \oplus_{\mathcal{W}} y &= (v_1 \oplus_{\mathcal{V}} v_2, u_1 \oplus_{\mathcal{U}} u_2) \text{ for any } x = (v_1, u_1), \ y = (v_2, u_2) \in \mathcal{W} \\ \alpha \odot_{\mathcal{W}} x &= (\alpha \odot_{\mathcal{V}} v, \alpha \odot_{\mathcal{U}} u) \text{ for any } x = (v, u) \in \mathcal{W} \text{ and any } \alpha \in \mathbb{F} \end{aligned}$$

Answer the following:

(a) Show that, with these operations, $\mathcal{W}$ is a vector space over $\mathbb{F}$

(b) If $\mathcal{B}_{\mathcal{V}} = v_1, v_2, \cdots, v_m$ is a basis for $\mathcal{V}$ and $\mathcal{B}_{\mathcal{U}} = u_1, u_2, \cdots, u_n$ is a basis for $\mathcal{U}$, find a basis for $\mathcal{W}$

(c) True or False ?

$$dimension\ of\ \mathcal{W} \ = \ dimension\ of\ \mathcal{V}\ + dimension\ of\ \mathcal{U}$$