# Home Work 1

Due Date: 12th Sep 2010

(Attempt all questions)

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**Q. 1]** Let $\| \ \|$ denote the standard Euclidean norm on $\mathbb{R}^2$. For each of the following subsets of $\mathbb{R}^2$ say whether it is closed / open / neither. Also identify the sets which are convex / bounded. For each set give explaination for your answer.

a) $\{ \ v \in \mathbb{R}^2 \ : \ \|v\| < 1 \ \}$
b) $\{ \ v \in \mathbb{R}^2 \ : \ \|v\| = 1 \ \}$
c) $\{ \ v \in \mathbb{R}^2 \ : \ \|v\| \leq 1 \ \}$
d) $\{ \ v \in \mathbb{R}^2 \ : \ 1 < \|v\| \leq 2 \ \}$
e) $\{ \ (x_1, x_2) \ : \ 5x_1 + 2x_2 + 3 = 0 \ \}$
f) $\{ \ (x_1, x_2) \ : \ 3x_1 + x_2 < 4 \ \}$
g) $\{ \ (x_1, x_2) \ : \ x_1 + 2x_2 \leq 3 \ \}$
h) $\{ \ (x_1, x_2) \ : \ -1 < x_1 < 1 \ \}$
i) $\{ \ (x_1, x_2) \ : \ -1 < x_1 < 1, \ x_2 = 0 \ \}$
j) $\{ \ (x_1, x_2) \ : \ 0 \leq |x_1| + |x_2| < 2 \ \}$

5 marks

**Q. 2]** Prove that union / intersection of two closed sets is closed. Can we make similar conclusion for bounded sets or convex sets?

5 marks

**Q. 3]** Plot the following functions and see if there exists any local minima. State the necessary and sufficient conditions for existance of local minima and verify your earlier observations using that.

a) $f(x, y) = 3x^2 + 5y^2, \quad -1 \leq x \leq 1, -1 \leq y \leq 1.$
b) $f(x, y) = 3x^2 + 5y^3, \quad -1 \leq x \leq 1, -1 \leq y \leq 1.$
c) $f(x, y) = 3x^4 + 5y^4, \quad -1 \leq x \leq 1, -1 \leq y \leq 1.$
d) $f(x, y) = -3x^2 - 5y^2, \quad -1 \leq x \leq 1, -1 \leq y \leq 1.$

5 marks

**Q. 4]** Plot the following functions and see if they are convex. Also observe the existance of local / global minima.

a) $f(x) = 2 + 3x$ on $\mathbb{R}$
b) $f(x) = |x|$ on $\mathbb{R}$
c) $f(x) = -|x|$ on $\mathbb{R}$
d) $f(x) = \exp(x^2)$ on $\mathbb{R}$
e) $f(x) = \exp(-x^2)$ on $\{x \,|\, x \geq 10\}$ 5 marks

**Q. 5]** Plot the following function and mark all the local minima / maxima.

$$f(x) = x^3 + x^2 + x + \frac{10^3 x}{1 + x^2}, \quad x \in [-10, 10].$$

• Analytically verify that there exists local maxima / minima for the above function in the following intervals: $[-5, -3]$ , $[-2, 0]$, $[0, 2]$, $[3, 5]$.

5 marks

**Q. 6]** Consider the problem of minimizing the linear form $f(x, y) = 0.1x + y$ on the 2D plane over the triangle with the vertices $(1, 0)$, $(0, 1)$, $(0, 0.5)$ (draw the picture!). Does the minimizer exist? Is it unique

5 marks

**Q. 7]** Let $A$ be a $10 \times 10$ symmetric matrix. From linear algebra we know that $A$ can expressed as: $A = \sum_{i=1}^{10} \lambda_i \, v_i \, v_i^\top$, where $\lambda_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^{10\times 1}$ such that $v_i^\top v_j = 0$ if $i \neq j$ and 1 if $i = j$. Show that the following statements are equivalent:
a) $A$ is positive semidefinite i.e. $u^\top A u \geq 0$ for all $u \in \mathbb{R}^{10\times 1}$.
b) $\lambda_i \geq 0$ for $i = 1, \ldots, 10$.
c) $A = B^2$ for certain matrix $B \in \mathbb{R}^{10\times 10}$.
d) $A = D^\top D$ for certain matrix $D \in \mathbb{R}^{10\times 10}$.

• What can you say about the minimum and maximum value of the following:
(i) $f(x) = x^\top A x$ on whole $\mathbb{R}^{10}$ ( Analyze when $\lambda_i$'s are positve / negative / both)
(ii) $f(x) = x^\top A x$ on $\{x \in \mathbb{R}^{10} : \|x\| = 1\}$

• Show that sum of the diagonal entries of $A$ is equal to $\sum_{i=1}^{10} \lambda_i$.

10 marks

**Q. 8]** Derive conditions on $A$ such that $f(x) = \frac{1}{2}(x - x_0)^\top A(x - x_0)$ is coercive. Note that $A \in \mathbb{R}^{n\times n}, x, x_0 \in \mathbb{R}^n$. If $f$ is coercive compute its global minimum. Is the function $f(x) = (c^\top x)^2$ a coercive function? Give reasons. $c, x \in \mathbb{R}^n$. 10 marks